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Journal of Electrostatics 66 (2008) 142–146
Journal of ELECTROSTATICS
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# Estimation of capacitance of different conducting bodies by the method of rectangular subareas

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Received 15 September 2006; received in revised form 29 October 2007; accepted 13 November 2007
Available online 26 December 2007

## Abstract

The capacitance evaluation of arbitrary-shaped conducting bodies is an important step for the estimation of spacecraft equivalent circuit model for the prediction of electrostatic discharge. In this paper, an attempt has been made for the evaluation of charge distribution and hence the capacitance of arbitrary-shaped conducting surfaces. Surfaces are modeled by planar rectangular subdomains in which the charge density is assumed to be constant. The exact formulation for the matrix element is evaluated for rectangular subsection. The Method of Moments with pulse basis function and point matching is employed to calculate the charge distribution on the surface and hence the capacitance. This paper presents the results for capacitance of different conducting shapes, e.g., square, rectangular, circular, annular circular disk, T-shaped, L-shaped, triangular, annular triangular, etc. The results have been compared with other available results in literature wherever possible.


*Keywords:* Electrostatic discharge; Electrostatic capacitance; Conductors; Rectangular subdomain; Method of Moments

## 1. Introduction

There has been considerable interest in the evaluation of the capacitance and charge distribution of different conducting structures such as rectangular plates, square plates, circular and annular discs, hollow cylinder, etc., located in free space because their use in spacecraft. The capacitance evaluation has an important application for the determination of spacecraft equivalent circuit models for the prediction electrostatic discharge. The earliest work on the evaluation of capacitance of square plate appears to have been carried out by Maxwell [1]. Smythe had derived closed form/approximate expressions for computing capacitance of a few conducting objects such as bowl, cylinder, circular disc, etc. [2]. There are some interesting publications on the capacitance calculation of three-dimensional multiconductor systems and different high voltage electrode configurations [3–4]. In the work of Ruehli and Brennan, the basic equations for the potential coefficients of rectangular conducting element were derived and used for the evaluation of capacitance of square plate, cube via Method of Moments [3]. However, the resulting equations for the potential coefficients are found to be complicated and also these were mainly used for two-/three-dimensional bodies with square/rectangular surfaces [3]. In Ref. [4], the capacitance to ground of different high voltage electrode configurations was evaluated, neglecting the influence of other grounded or live structures. Also different methods of calculation were used for different electrode geometries, e.g., sphere, toroid, cylinder, circular disc, rectangular plate, and box [4]. However, the authors had not noticed any work on the evaluation of capacitance of arbitrary planar conducting bodies with a more generalized and simple elemental shape, which can be used for any planar surface. Harrington evaluated data on the capacitance of a square conducting plate employing square subdomain regions, but did not present clearly the exact formulas for

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0304-3886/$ - see front matter 
doi:10.1016/j.elstat.2007.11.003

the evaluation of the matrix elements for the evaluation of capacitance [5]. The triangular subdomains had been used for more complex surfaces by Rao et al. [6]. Also, Chakraborty et al. [7] had obtained a method of computing the capacitance of a cylinder and a truncated cone by employing cylindrical subsections. In the present paper, the authors have concentrated on the numerical evaluation of capacitance of conducting objects in free space using Method of Moments and rectangular subdomain modeling. The rectangular shape is chosen because of its ability to conform easily to any geometrical surface or shape and at the same time maintaining the simplicity of approach compared to the triangular patch modeling. Here, the exact formulation for the evaluation of the impedance matrix for rectangular subdomain is determined. The results are compared with other available data in literature.

## 2. Formulation

We consider a perfectly conducting surface is charged to a potential $V$. The unknown surface charge density distribution $\sigma(r')$ may then be determined by solving the following integral equation:

$$V = \int\int_S \frac{\sigma(r')}{4\pi\varepsilon|r-r'|}\,\mathrm{d}s'. \tag{1}$$

Here, $r$ and $r'$ are the position vectors corresponding to observation and charge source points, respectively, $\mathrm{d}s'$ is an element of surface $S$ and $\varepsilon$ is the permittivity of free space. The exact solution for the charge distribution can be obtained only for a few very specialized geometries. In the general case, the surface is discretized and the charge distribution is found by solving Eq. (1) using numerical methods. Here, the arbitrary-shaped bodies are approximated by planar rectangular subdomains (Fig. 1). The Method of Moments with pulse basis function and point matching is then used to determine the approximate charge distribution. On each subdomain, a pulse expansion function $P_n(r)$ is chosen such that $P_n(r)$ is equal to 1 when $r$ is in the $n$th rectangle and $P_n(r)$ is equal to 0 when $r$ is not in the $n$th rectangle. With the above definition of expansion function, the charge density, $\sigma(r')$ may be approximated as follows:

$$\sigma(r') = \sum_{n=1}^{N} \sigma_n P_n(r') \quad \text{where } P_n = \begin{cases} 1 & \text{for } n\text{th subsection} \\ 0 & \text{elsewhere} \end{cases} \tag{2}$$

Here, $N$ is the number of rectangles modeling the surface and $\sigma_n$'s are the unknown weights (charge density).

Substitution of charge expansion (2) in (1) and point matching the resulting functional equation, by enforcing equality of the two sides of the equation for observation points located at the center of each rectangle, yields an $N \times N$ system of linear equations which may be written in the following form

$$[V] = [K][Q]. \tag{3}$$

Here, $[K]$ is an $N \times N$ matrix and $[Q]$ and $[V]$ are column vectors of length $N$.

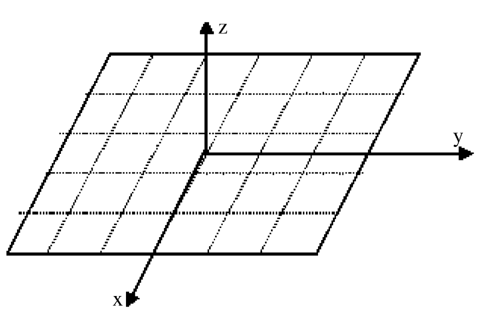


Fig. 1. Square plate divided into rectangular subsections.

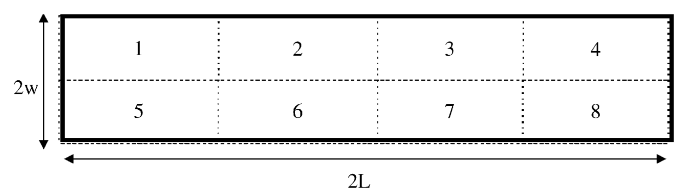


Fig. 2. Rectangular plate ($2L = 4$ m; $2w = 1$ m; $V = 1$ V) divided into $4 \times 4$ subsections; capacitance = 54.73 pF.

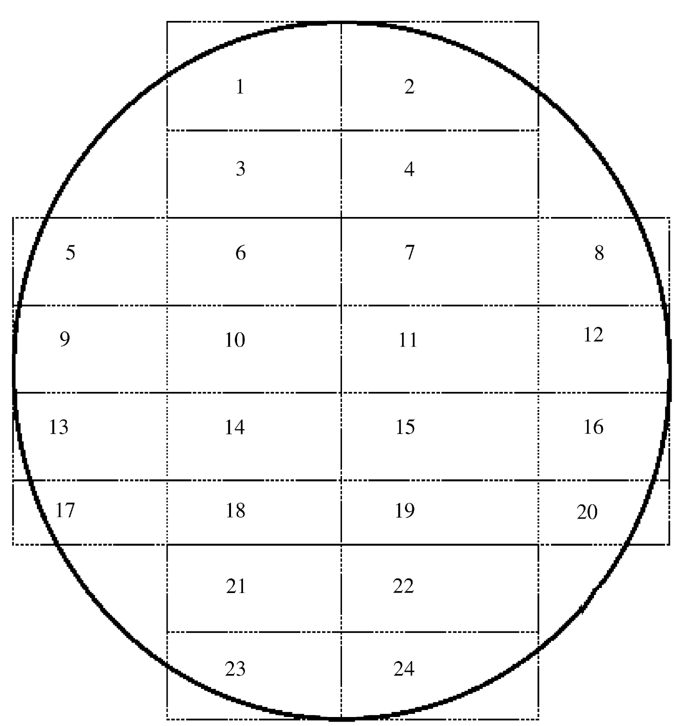


Fig 3. Circular disc (radius = 1 m, $N = 24$); capacitance = 68.36 pF agrees with analytical value = 70.73 pF [11].

The elements of [K], [Q], and [V] are given as follows:

$$K_{mn} = \iint\limits_{\text{rectangle}} \frac{1}{4\pi\varepsilon |r_m - r'|} \mathrm{d}A'$$

$$Q_n = \sigma_n = \text{unknown charge density in subdomain } n$$

$$V_n = V \tag{4}$$

where $r_m$ denotes the position vector of the center of the $m$th rectangle. $A'$ is the area of the source rectangle.

$$|r_m - r'| = \sqrt{(x_m - x')^2 + (y_m - y')^2}.$$

Here, we have considered the conducting surface at $z = 0$ plane.

Since the numerical formulation of (1) via the Method of Moments is well known [5], we consider only the evaluation of the element of the moment matrix as given by Eq. (4). Each element corresponds to the potential at some point in space, $r = (x, y, z)$, due to a rectangular patch of surface charge of unit charge density. In general, the patch is arbitrarily positioned and oriented in space.

The integration of Eq. (4) is quite tedious, but the final result is relatively simple [8].

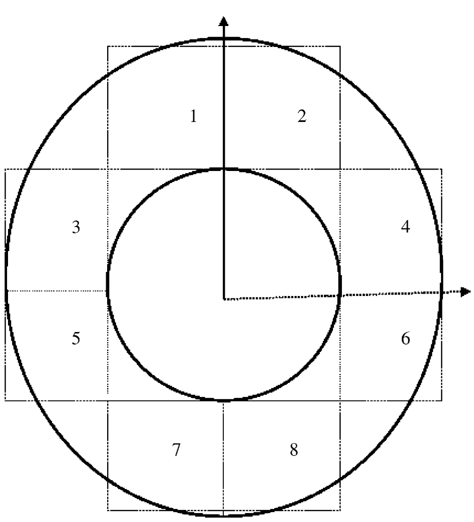


Fig. 4. Annular disk (inner radius = 1 m, outer radius = 2 m, $N = 8$); capacitance = 33.86 pF agrees with Ref. [12].

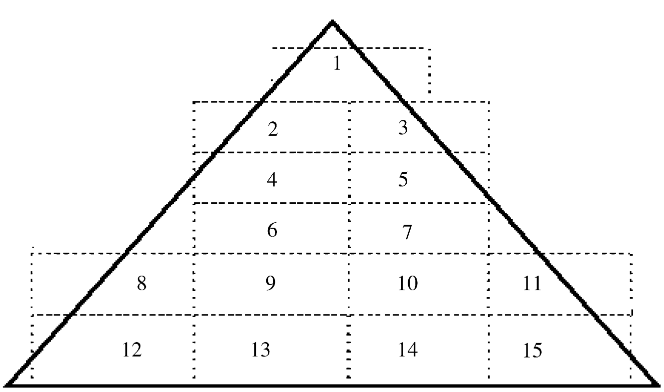


Fig. 5. Equilateral triangular plate with sides each of 1 m divided into 15 subsections; capacitance = 23.1 pF.

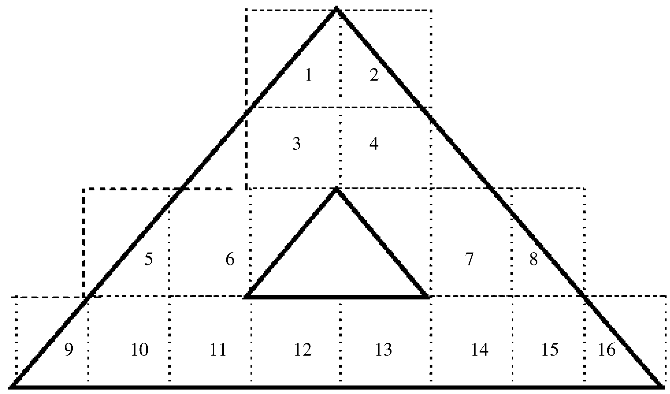


Fig. 6. Annular triangular plate with sides of 1 m and inner triangle with sides of 0.25 m divided into 16 subsections; capacitance = 26.97 pF.

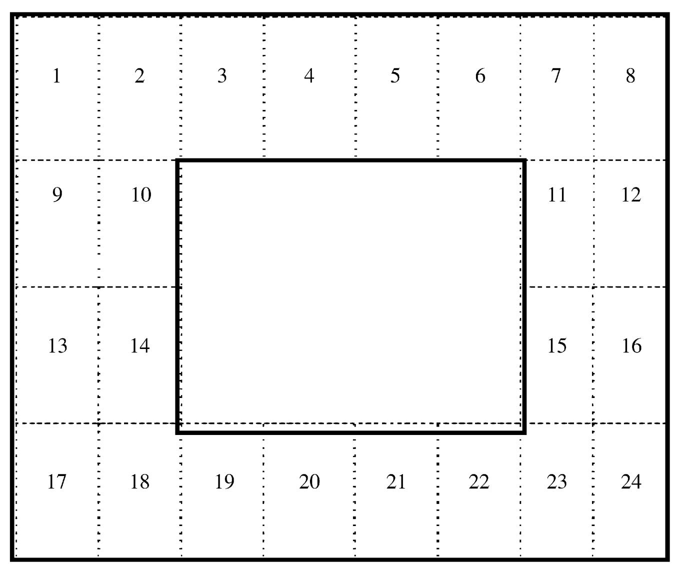


Fig. 7. Annular square (side = 1 m, side of annulus = 0.5 m, no. of subsections = 24, $C = 36.66$ pF).

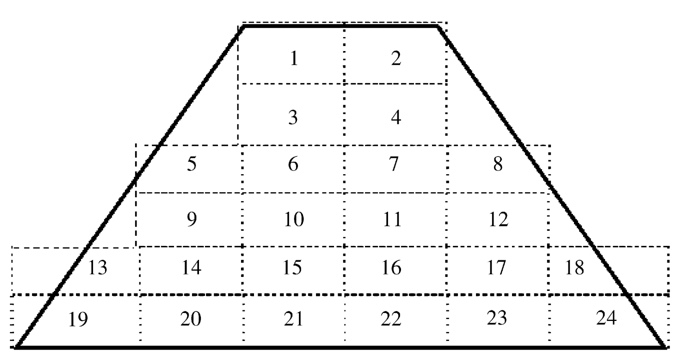


Fig. 8. Trapezoidal plate (sides = 1, 0.6, 0.3 m; $N = 12$; $C = 20.41$ pF).

For the diagonal elements of the matrix, the integration is evaluated as follows:

$$K_{nn} = \frac{1}{\pi\varepsilon}\left(a\ \ln\left(\frac{b}{a}+\sqrt{\frac{b^2}{a^2}+1}\right)+b\ \ln\left(\frac{a}{b}+\sqrt{\frac{a^2}{b^2}+1}\right)\right). \tag{5}$$

Here, $a$ and $b$ are the sides of each rectangular subsection.

Using the standard integral formula the non-diagonal elements are evaluated as follows:

$$K_{mn} = -\left[\left|x_m - x'\right| \ln \frac{\left(\left|y_m - y_n + b\right| + \sqrt{(x_m - x')^2 + \left(y_m - y_n + b\right)^2}\right)}{\left(\left|y_m - y_n - b\right| + \sqrt{(x_m - x')^2 + \left(y_m - y_n - b\right)^2}\right)}\right]_{x_n - a}^{x_n + a}$$
$$-\left[\left|y_m - y'\right| \ln \frac{\left(\left|x_m - x_n + a\right| + \sqrt{\left(y_m - y'\right)^2 + (x_m - x_n + a)^2}\right)}{\left(\left|x_m - x_n - a\right| + \sqrt{\left(y_m - y'\right)^2 + (x_m - x_n - a)^2}\right)}\right]_{y_n - b}^{y_n + b} \tag{6}$$

Here, the source point is $(x_n, y_n)$ and the field point is $(x_m, y_m)$. The $x'$ and $y'$ of Eq. (6) are replaced by their respective limits. Solution of the matrix Eq. (3) yields values for the surface charge density at the centers of the subdomains. The capacitance, $C$, of the body is obtained from the following equation:

$$C = \frac{Q}{V} = \frac{1}{V}\sum_{n=1}^{N} \sigma_n A_n. \tag{7}$$

Here, $N$ is the total number of rectangular subsections.

## 3. Numerical results and discussions

A computer program based on the preceding formulation has been developed to determine the charge distribution and hence capacitance for the following conducting surfaces: square plate, rectangular plate, T-shaped plate, L-shaped plate, circular disc, annular circular disc, trapezoidal plate, triangular plate, annular triangular plate, etc. The corresponding figures are shown in Fig. 2–10. The

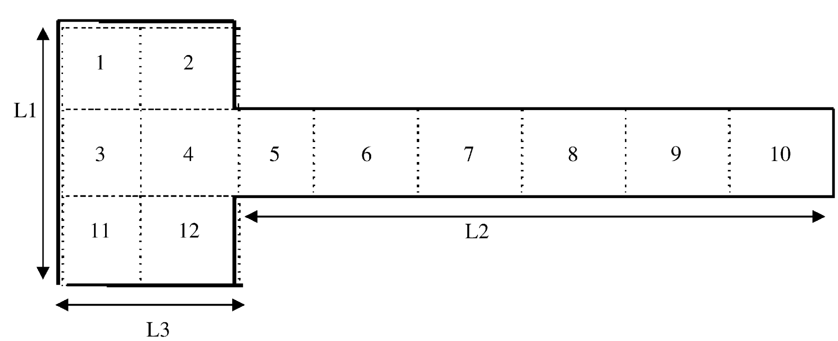


Fig. 9. T-shaped plate (L1 = 3 m; L2 = 3 m; L3 = 1 m; $N$ = 12; $C$ = 98.53 pF).

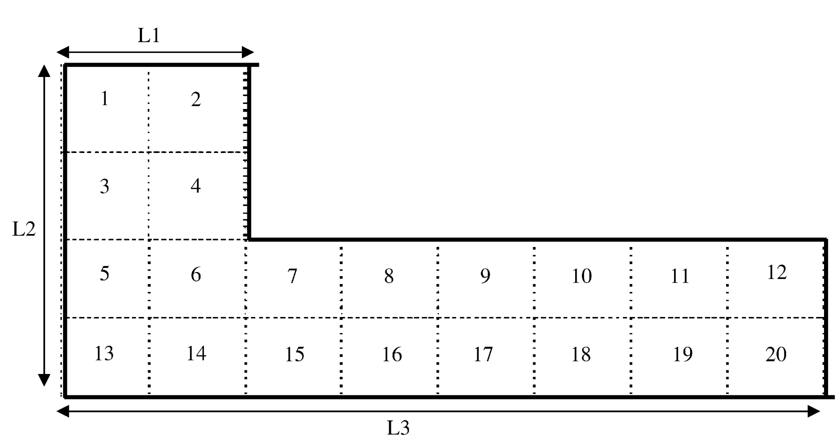


Fig. 10. L-shaped plate (L1 = 1 m; L2 = 2 m; L3 = 4 m; $N$ = 20; $C$ = 94.04 pF).

results have been compared with other available results in literature. The result for a rectangular plate agrees with the available data in literature [9–11]. The result for a circular disc (radius = 1 m, $N = 24$, capacitance = 68.36 pF) agrees with analytical value = 70.73 pF [12]. Also the capacitance of an annular disk (inner radius = 1 m, outer radius = 2 m, $N = 8$, $C = 33.86$ pF) shows well matching with the data available in literature [13].

## 4. Conclusion

A simple and efficient numerical procedure is presented for treating electrostatic problem involving complex geometrical shaped planar conducting bodies. This method of capacitance evaluation by dividing the conducting surface into rectangular subsection may be extended for non-planar geometrical bodies, e.g., satellite structure. Also this method may find important application for the determination of equivalent circuit models of multiconductor or multiwire arrangements used in electronic systems.

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