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SCIENCE DIRECT
C. R. Mecanique 333 (2005) 127–132
COMPTES RENDUS MECANIQUE
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An example of entropy balance in natural convection, Part 1: the *usual* Boussinesq equations

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Received 6 April 2004 ; accepted after revision 28 October 2004

Available online 8 January 2005

Presented by Sébastien Candel

This work is dedicated to the memory of Bernard Spinner

Abstract

Numerical simulations of natural convection in cavities performed with the *usual* Boussinesq equations result in an unbalanced irreversibility budget. Thermodynamic analysis shows that these equations represent a system that exchanges with the surroundings, not only two heat fluxes, but also two fluxes of mechanical energy: an input, that generates the fluid motion, and an output, due to viscous friction. After this analysis, the thermodynamic discrepancies can be explained. ***To cite this article: M. Pons, P. Le Quéré, C. R. Mecanique 333 (2005).***

Résumé

Un exemple de bilan d'entropie pour la convection naturelle, Partie 1 : les équations de Boussinesq *usuelles*. Les simulations numériques réalisées avec les équations de Boussinesq *usuelles* ne peuvent pas donner un bilan d'irréversibilités fermé. L'analyse thermodynamique démontre que ces équations représentent un système qui échange avec l'extérieur, en plus des deux flux de chaleur, deux flux d'énergie mécanique : un flux entrant, qui est la source du mouvement du fluide, un flux sortant, qui est dû à la friction visqueuse. Grâce à cette analyse, les incohérences thermodynamiques trouvent une explication. ***Pour citer cet article : M. Pons, P. Le Quéré, C. R. Mecanique 333 (2005).***

Keywords: Heat transfer; Fluid mechanics; Natural convection; Boussinesq; Thermodynamics; Irreversibility

Mots-clés : Transferts thermiques ; Mécanique des fluides ; Convection naturelle ; Boussinesq ; Thermodynamique ; Irréversibilités

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1631-0721/$ – see front matter
doi:10.1016/j.crme.2004.11.011

1. Introduction

For reasons of publication, the work presented here is divided in two parts, this article (Part 1) and Part 2 [1].

The Boussinesq approximation of the Navier–Stokes equations is more than a century old [2,3]; it is presented in many textbooks or articles [4–9] where its validity is also discussed. These equations are extensively used for modelling natural convection, except when the temperature difference is very large, in geophysical configurations, and for near-critical fluids. About second law analysis of natural convection, one-phase pure fluids present only two sources of irreversibility, heat-diffusion and viscous friction. The literal expressions of these irreversibilities as functions of velocity and temperature can be found in many textbooks, for instance [4–6,10,11]. Second law analyses of natural convection are rather scarce. Most of the time, they consist of calculating the two fields of irreversibility in their respective non-dimensional scales. However, establishing a balanced entropy budget requires giving all the irreversibilities in a common scale. In Section 2, such a thermodynamic analysis, very simple, is applied to buoyancy-driven natural convection with two heat sources. It leads to thermodynamic conditions fulfilled by any real system, and that should therefore be respected by any good numerical simulation. In Section 3, are presented numerical results obtained with the *usual* Boussinesq (UB) equations in one very simple case corresponding to the famous benchmark of De Vahl Davis [12,13]. The contradictions with the second law presented by the numerical solutions make a thermodynamic analysis of the UB equations necessary.

2. Thermodynamic milestones about natural convection

The thermodynamic analysis developed in this section applies to *any real system* in the following conditions: buoyancy-driven natural convection in steady-state with two rigid impermeable walls at fixed temperatures, one at T_h the other at T_c ($T_h > T_c$; the indexes h and c stand for the *hot* and *cold* walls), with the two other boundaries adiabatic, rigid and closed (or ruled by symmetry). This includes the Rayleigh–Bénard configuration (either with adiabatic vertical walls or in infinite geometry), the differentially-heated cavity with adiabatic horizontal walls, etc. The fluid is assumed to be one-component one-phase and at local thermodynamic equilibrium.

2.1. First law analysis of steady-state

The heat fluxes through the hot and cold walls are respectively Q_h (positive) and Q_c (negative). As the barycentre is fixed (because of stationarity) and the walls closed, rigid and immobile, the system does not exchange any work with the surroundings, but only these two heat fluxes. The first law in steady-state readily writes: $Q_h + Q_c = 0$. Taking as a reference the heat flux Q_λ that would be exchanged between the active walls by pure conduction through the fluid at rest, the respective Nusselt numbers are defined: $Nu_h = Q_h/Q_\lambda$, and $Nu_c = -Q_c/Q_\lambda$. The non-dimensional form of the first law is: $Nu = Nu_h = Nu_c$. This first thermodynamic milestone is obvious. Looking now in some more detail, the fluid follows a convection loop in the cavity, with the following ingredients: kinetic energy, shear in the boundary layers, and viscous friction, i.e. dissipation of kinetic energy into heat. In steady-state, the global kinetic energy of the fluid obviously remains constant. As a consequence, the continuous viscous dissipation (W_v) is exactly compensated by a continuous generation of mechanical energy (W_m) in the system itself, see for instance [5], p. 111, or [6], p. 179. Non-dimensionalising W_m and W_v by the reference heat flux Q_λ, that equality becomes: $N_{Wm} = W_m/Q_\lambda = W_v/Q_\lambda = N_{Wv}$. This is the second thermodynamic milestone. From this energetic description, the real system is schematically represented by the energy diagram, Fig. 1(a).

From another point of view, the system exchanges heat fluxes with two heat sources, one hot one cold, and produces mechanical energy. Like for any heat-engine, a conversion efficiency can be defined according to: $\eta = W_m/Q_h = N_{Wm}/Nu_h$.

2.2. Second law analysis of steady-state

When considering the system in steady-state as a whole, the rate of total entropy production is: $\Sigma = Q_h(T_c^{-1} - T_h^{-1})$. Since energy fluxes are non-dimensionalised by the heat flux in a purely conductive system, all the quantities involving entropy are non-dimensionalised by the entropy production Σ_λ that would exist in that same system: $\Sigma_\lambda = Q_\lambda(T_c^{-1} - T_h^{-1})$. It is easily proved that, in steady-state, the number of total irreversibility N_Σ always equates the Nusselt number:

$$N_\Sigma = \frac{\Sigma}{\Sigma_\lambda} = \frac{Q_h(T_c^{-1} - T_h^{-1})}{Q_\lambda(T_c^{-1} - T_h^{-1})} = \frac{Q_h}{Q_\lambda} = Nu_h$$

Let us denote by $N_{\Sigma q}$ and $N_{\Sigma v}$ the numbers of irreversibility corresponding to the entropy productions respectively by heat diffusion and by viscous friction, non-dimensionalised as described above. The equality above can then be written:

$$N_{\Sigma q} + N_{\Sigma v} = Nu \tag{1}$$

In the steady-state, the sum of the different non-dimensional irreversibilities (herein heat-diffusive and viscous) *must equate the Nusselt number*. This is the third thermodynamic milestone; in the limits of our knowledge it was never mentioned before. Another expression of the second law says that the conversion efficiency of any heat engine operated between two heat sources at T_h and T_c is less than the Carnot efficiency: $\eta_C = (T_h - T_c)/T_h$. Any real system necessarily fulfils: $\eta < \eta_C$. This inequality is a fourth thermodynamic milestone.

3. Numerical results with the *usual* Boussinesq system

Let us now check whether numerical calculations done on a very simple example chosen among the systems analysed in Section 2 are in agreement with those four thermodynamic milestones. The numerical example simulates one two-dimensional differentially-heated cavity filled with air and in steady-state (see the benchmark of De Vahl Davis). Let us first describe the numerical model.

3.1. *Non-dimensional* usual *Boussinesq system*

The UB equations are well-known and will not be recalled. The fluid density is assumed to depend only on temperature: $\rho = \rho_0[1 - \beta(T - T_0)]$, where β is the thermal expansion coefficient, and the state 0 defined by the average temperature $T_0 = (T_h + T_c)/2$. Those equations are non-dimensionalised by taking as references (1) the height of the fluid domain H for distances, (2) the speed $V^* = (\alpha/H)Ra^{1/2}$ for velocities, (3) the temperature difference $\Delta T = T_h - T_c$ for the difference $T - T_0$. The Rayleigh number is $Ra = (\beta g H^3 \Delta T)/(\nu\alpha)$, α is the fluid thermal diffusivity, g is the gravity acceleration, ν is the fluid kinematic viscosity. In addition to the Rayleigh number, the control parameters are the aspect ratio A_r (height/width=H/L) and the fluid Prandtl number Pr. With the non-dimensional quantities, coordinates x and z, velocity components u and w, time τ, pressure Π, and temperature θ, and in the framework of the Fourier and Newton laws, the non-dimensional equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

$$\begin{aligned} &\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{\partial \Pi}{\partial x} + \frac{Pr}{Ra^{1/2}}\nabla^2 u \\ &\frac{\partial w}{\partial \tau} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial \Pi}{\partial z} + \frac{Pr}{Ra^{1/2}}\nabla^2 w + Pr\,\theta \end{aligned} \tag{3}$$

$$\frac{\partial \theta}{\partial \tau} + u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = \frac{1}{Ra^{1/2}}\nabla^2 \theta \tag{4}$$

The boundary conditions are: no slip at the four walls; $\theta = +0.5$ at $x = 0$ and $\theta = -0.5$ at $x = x_c \equiv 1/A_r$; adiabaticity for the two walls at $z = 0$ and $z = 1$. The numerical model used for this study has been formerly developed in LIMSI and is described in details in [14].

3.2. Thermodynamic balances

The Nusselt numbers are:

$$Nu_h = \frac{Q_h}{Q_\lambda} = \frac{-1}{A_r}\int_0^1 \left(\frac{\partial\theta}{\partial x}\right)_{x=0} \mathrm{d}z; \qquad Nu_c = \frac{Q_c}{-Q_\lambda} = \frac{-1}{A_r}\int_0^1 \left(\frac{\partial\theta}{\partial x}\right)_{x=x_c} \mathrm{d}z \tag{5}$$

The UB equations ensure their equality, so it is included in our calculations. The irreversibility balance (1) also involves the numbers of conductive and viscous irreversibility, obtained by integration of the corresponding entropy productions over the whole fluid volume and adequate non-dimensionalisation:

$$N_{\Sigma q} = \frac{1}{A_r}\int_0^1\int_0^{x_c} \frac{(\partial\theta/\partial x)^2 + (\partial\theta/\partial z)^2}{(1+\theta\,\Delta T/T_0)^2}\,\mathrm{d}x\,\mathrm{d}z \tag{6}$$

$$N_{\Sigma v} = \frac{2}{A_r}\frac{\beta g H}{C_p}\frac{T_0}{\Delta T}\int_0^1\int_0^{x_c} \frac{(\partial u/\partial x)^2 + \frac{1}{2}(\partial u/\partial z + \partial w/\partial x)^2 + (\partial w/\partial z)^2}{1+\theta\,\Delta T/T_0}\,\mathrm{d}x\,\mathrm{d}z \tag{7}$$

It must be here noticed that, among the three quantities involved in Eq. (1), $N_{\Sigma v}$, and only $N_{\Sigma v}$, involves a fourth non-dimensional parameter: $\beta g H T_0/(C_p \Delta T)$, where C_p is the fluid specific heat at constant pressure. This parameter, independent of A_r, Pr, and Ra, and that does arise neither from the UB equations nor from the boundary conditions, is denoted by ϕ in the followings. Bejan [5,10] and Gebhart et al. [4] point out that ϕ might easily be comparable to one, both authors concluding that the viscous irreversibility is *not necessarily negligible*.

3.3. Numerical results

The numerical results presented here-under are obtained with air at $T_0 = 300$ K ($Pr = 0.71$) in a square cavity ($A_r = 1$), with $Ra = 10^6$. Because of the fourth parameter ϕ, either the height H or the temperature difference ΔT must be fixed. We have chosen 1.2866 m as the cavity height H, which results in a temperature difference ΔT of 4.890 mK. These values emphasize as much as possible the effect we mean to highlight, while preventing the results from physical aberration (see discussion below). It must also be mentioned that such a cavity size belongs more to building engineering than to geophysics, and that such a small ΔT validates the assumptions of solenoidal flow and constant fluid properties. The present problem differs absolutely from that of large temperature differences and non-Boussinesq flows. Lastly, this is a numerical experiment, not a physical one. The results are presented in Table 1. The total irreversibility number, $N_\Sigma = N_{\Sigma q} + N_{\Sigma v}$, is found strictly larger than the Nusselt number, in contradiction with Eq. (1). Moreover, the equality obtained is: $N_{\Sigma q} = Nu$. These two features are found for any calculation done with the UB equations, which therefore must be revisited from the point of view of thermodynamics.

3.4. *Thermodynamic analysis of the* usual *Boussinesq equations*

The UB equations are herein strictly considered for themselves, no longer as approximations of the Navier–Stokes equations, and simple thermodynamic consequences are drawn. First, a divergence-free flow means that the fluid is incompressible. Second, applying the momentum equation (3) to an incompressible fluid means that the fluid is locally submitted to a vertical force by which the fluid motion is induced. Thermodynamically speaking,

Table 1
Thermodynamic balances calculated with the *usual* Boussinesq equations, for a cavity height H of 1.2866 m at $Ra = 10^6$. One has $\phi = 2.568$, and $\Delta T/T_0 = 16.30 \times 10^{-6}$

Boussinesq model	Nu	$N_{\Sigma q} + N_{\Sigma v}$	$N_{\Sigma q}$	$N_{\Sigma v}$	N_{Wm}
Usual	8.8407	17.6431	8.8407	8.8024	143.48×10^{-6}

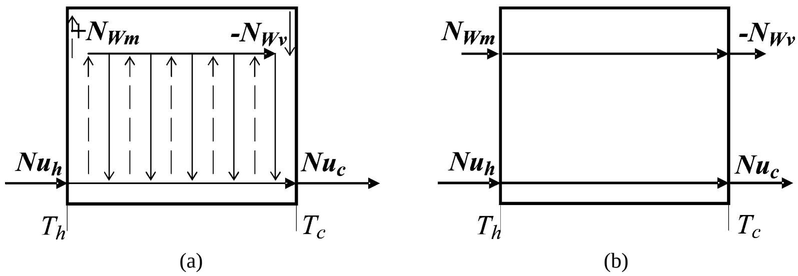

Fig. 1. Energy diagram describing natural convection in steady-state. The lower level represents thermal energy, the upper level mechanical energy. The fluxes of thermal and mechanical energies are indicated in non-dimensional form ($Nu_h, Nu_c, N_{Wm}, N_{Wv}$). For sake of clarity, the heat generated by viscous friction is not mentioned. (a) Real case: the production of mechanical energy inside the system is symbolised by the upward dashed arrows, the viscous dissipation by the downward solid arrows. (b) System simulated with the *usual* Boussinesq equations: the two heat fluxes are equal, but the fluid also receives mechanical energy, and viscous dissipation corresponds to transfer of work; internal exchanges between work and heat are discarded.

the fluid receives mechanical energy, with a non-dimensional local density in $(\theta - \theta^*)w$, where θ^* is an arbitrary but constant non-dimensional temperature. The integral of that energy density is W_m. In non-dimensional form, and noticing that the integral of w is zero, one has:

$$N_{Wm} = \frac{Ra^{1/2}}{A_r}\frac{\beta g H}{C_p}\int_0^1\int_0^{x_c}(\theta w)\,\mathrm{d}x\,\mathrm{d}z \tag{8}$$

As θ and w are mostly of same sign, N_{Wm} is positive. Now, as third point, where does this mechanical energy come from? In the real world, the system only exchanges heat fluxes, the mechanical energy responsible of the fluid motion is thus surely generated within the fluid, and this process, for sake of total energy conservation, consumes some internal energy, see Fig. 1(a). In the numerical simulation however, the heat equation (4) only accounts for advection and diffusion, with no source-term that could transform heat into mechanical energy: in other words the simulated N_{Wm} does not come from the fluid, it comes from outside. This analysis shows that the UB equations implicitly assume that the *forcing* term of the momentum equation represents an external force, and that mechanical energy is consequently supplied to the simulated system by its surroundings. This implicit assumption is in contradiction with the real world. Now, as fourth point, the total non-dimensional kinetic energy lost by viscous friction is:

$$N_{Wv} = \frac{-1}{A_r}\frac{\beta g H}{C_p}\int_0^1\int_0^{x_c}(u\nabla^2 u + w\nabla^2 w)\,\mathrm{d}x\,\mathrm{d}z \tag{9}$$

In the real world, the energy dissipated transforms into heat, an irreversible process. In the simulated system however, the viscous term of the momentum equation has no counterpart in the heat equation. Some authors liken this feature to the presence of molecular-size turbines distributed in the fluid and that would withdraw from the fluid as much kinetic energy as lost in friction. In such a system, the kinetic energy lost in friction is not transformed into heat but released as work toward the surroundings. This point has two aspects. First, the simulated system is finally described by the energy diagram Fig. 1(b): in addition to the heat fluxes, this system exchanges with the surroundings two equal and opposite fluxes of mechanical energy while completely discarding the internal exchanges between heat and work. Second, that release of work is a reversible process, evidencing that viscous friction is not an irreversibility in the simulated system. It results from the latter observation that there remains only one source of irreversibility in the simulated system, heat diffusion, so that the irreversibility balance (1) reduces in $N_{\Sigma} = N_{\Sigma q} = Nu$. The numerical results agree with this equality. The last concern is the conversion efficiency η and its theoretical maximal limit η_{C}. From the values given in Table 1 one obtains: $\eta = 16.23 \times 10^{-6}$ and $\eta_{\mathrm{C}} = 16.30 \times 10^{-6}$. If the results of this calculation were a good approximation of reality, then the calculated values of Nu and N_{Wm}, and consequently η, would also be realistic, leading to a conversion efficiency very close to (but still less than) its maximal limit η_{C}. A real system producing irreversibility as reported in Table 1 cannot be that close to reversibility. We could have presented results obtained with $H = 1.29$ m and $\Delta T = 4.851$ mK (still with $Ra = 10^6$). The calculated values of η and η_{C} would have been respectively: 16.27×10^{-6} and 16.17×10^{-6}, in full contradiction with $\eta < \eta_{\mathrm{C}}$. The case presented herein just avoids this physical aberration, but shows that the UB equations do not prevent from violating the second law.

Actually, a careful reading of literature, [6,9] among others, shows that the case investigated herein should not be calculated with UB equations. Indeed, these authors indicate that the UB equations are valid when ϕ is small compared to 1. Herein, ϕ is larger than 2, so that this particular case should be approached with a heat equation called *extended* in Ref. [9], or *deep convection*, or *thermodynamic* in Ref. [7]. The *thermodynamic* Boussinesq model, the numerical results, and their thermodynamic analysis are the subject of the second part of this study [1].

4. Conclusion

The *usual* Boussinesq equations can never lead to balanced irreversibility budget. Indeed, they represent a system that, in addition to the heat fluxes exchanged with the hot and cold sources, also exchanges with the surroundings two fluxes of mechanical energy, equal and opposite. The input generates the fluid motion, the output replaces the generation of heat by viscous friction. Moreover, the *usual* Boussinesq model discards the internal exchanges between heat and work. As a first consequence, in the UB system viscous friction is not an irreversibility, and its irreversibility balance cannot be that of the real system. As a second consequence, the UB equations do not insure that the simulated system produces less work than allowed by the Carnot factor.

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