Available online at www.sciencedirect.com
ScienceDirect
Engineering Fracture Mechanics 75 (2008) 427–439
Engineering Fracture Mechanics
www.elsevier.com/locate/engfracmech

A weight function for 2D subsurface cracks under general loading conditions

M. Beghini [a], L. Bertini [a], V. Fontanari [b,*]

[a] Dip. di Ingegneria Meccanica, Nucleare e Produzione, University of Pisa, via Diotisalvi 2, 56126 Pisa, Italy
[b] Dip. di Ingegneria dei Materiali e delle Tecnologie industriali, University of Trento, via Mesiano 77, 38050 Trento, Italy

Received 8 December 2006; received in revised form 19 February 2007
Available online 27 February 2007

Abstract

A procedure for evaluating the fracture mechanics parameters of a subsurface two-dimensional crack parallel to the boundary in an elastic half plane is presented. A Weight Function (WF) with a matrix structure is proposed, to account for the coupling effects between modes I and II, typical in non-symmetrical problems. In order to face any loading condition, the WF was formulated by symmetric and anti-symmetric components and the 'multiple reference loading' approach was used to derive their analytical expressions. To this purpose, a parametric Finite Element (FE) analysis was set up and the Stress Intensity Factors (SIFs) were determined for several independent loading conditions. The analysis was carried out for different ratios between crack length and in-depth position and, consequently, the dependence of the WF on this parameter was studied. The WF accuracy was assessed by considering different loading and the method applied for evaluating the SIFs produced by a point-like load travelling on the semi-plane surface. The results indicated that the correct fracture mechanics analysis requires crack closure (either complete or partial) to be taken into account. Consequently, the crack opening displacement (COD) components under general loading conditions have to be evaluated. On the basis of the WF, the related Green Function (GF) was also derived by which the COD components can be efficiently evaluated for any applied load including the contact due to crack closure.

Keywords: Subsurface crack; Mixed mode; Weight function; Green's function; Crack closure

1. Introduction

Several mechanisms of failure in mechanical components, such as spalling or pitting in rolling contact [1–4] or nucleation of cracks in ultra high-cycle fatigue [5–7], are characterized by the initiation and growth of subsurface defects. The early stages of fatigue growth of a subsurface crack are generally characterized by mixed fracture modes. These phenomena have been studied by several authors in the framework of the fracture mechanics and many analyses have been carried out for determining the fracture mechanics parameters

* Corresponding author. Tel.: +39 0461 882430; fax: +39 0461 881977.
E-mail addresses: beghini@ing.unipi.it (M. Beghini), bertini@ing.unipi.it (L. Bertini), vigilio.fontanari@ing.unitn.it (V. Fontanari).

0013-7944/$ - see front matter
doi:10.1016/j.engfracmech.2007.02.017

[1–11]. If the distance of a crack like defect from the surface is sufficiently high compared to the defect dimension, the Griffith crack model can be assumed as representative. In contact fatigue, this approximation turns to be questionable as the observed subsurface cracks they have length comparable with the distance from the surface. The Finite Element (FE) method has been extensively applied to evaluate the Stress Intensity Factors (SIFs) under complex loading conditions [8] and to predict the crack growth paths [9–12]. However, FE analyses, since general and accurate, are significantly time consuming, particularly when the crack growth has to be predicted and many SIF calculations have to be performed under several loading conditions. The Weight Function (WF) method turns out to be suitable for solving this kind of problems.

The authors [13,14] have recently proposed a method for evaluating the fracture mechanics parameter for the two-dimensional subsurface crack under general loading based on the WF approach. A matrix like WF was adopted in order to account for the coupling effects between modes I and II due to the asymmetry induced by the presence of the free surface. Moreover, in order to consider the presence of the two crack tips, the WF was separated into symmetrical and anti-symmetrical components, thus allowing for a straightforward evaluation of the fracture mechanics parameters under completely general loading conditions. The 'direct adjustment method' [15] based on multiple reference loading cases was adopted to determine both diagonal and off-diagonal WF matrix components. To this purpose a FE parametrical analysis was carried out for different loading conditions and ratios between the crack length and its distance from the surface.

In the present paper an extended formulation of the WF is presented and assessed by comparison with SIFs calculated by accurate FE analysis. The proposed WF is intended to encompass a broader range of ratios between crack length and distance from the surface. The dependence of the WF on this geometrical parameter is studied thus giving quantitative indications about the effect of the free boundary on the problem. The WF was applied for determining the SIF histories induced at the tips of a subsurface crack by a point like load travelling on the surface. A parametrical analysis, including several ratios between crack length, distance from surface and inclinations of the travelling load, was carried out. This application intended to show the potentiality of the method in predicting the effects of a moving body in contact on the surface under general friction conditions.

Under this kind of mainly compressive loading, typical of the contact, complete or partial crack closure are expected. Therefore, a method for evaluating the COD components is necessary for a correct prediction of the FM parameters. To this purpose the Green Functions (GF) of the subsurface crack was obtained by a direct elaboration of the proposed WF. By the GF, the COD components can be efficiently calculated by direct integration under general loading conditions, and the effect of crack closure on the SIFs can be accurately accounted for by including the tractions between the crack faces due to closure.

2. Problem definition

An embedded 2D crack, having length $2a$, parallel to the surface of an elastic semi-plane and placed at a depth h is considered, as represented in Fig. 1. In order to provide an unambiguous definition of the fracture mechanics parameters for both the tips (in particular regarding the sign of K_{II} of which a general definition is not usually adopted), a Cartesian reference system is defined for each tip L and R. The tip reference systems

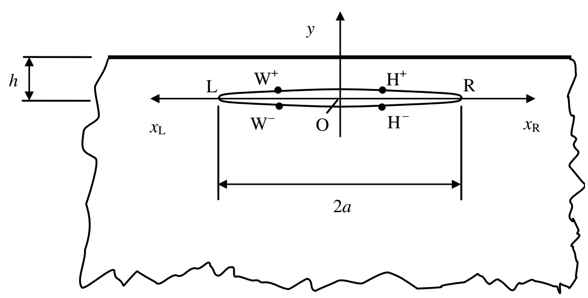

Fig. 1. Subsurface crack parallel to the semi-plane surface.

share the y-axis while they have opposite x-axes (x_L and x_R respectively) each of them having the positive direction of the crack growth for the related tip. The y-axis is the only axis of symmetry for the geometry, its versus is chosen toward the free surface. The relative displacements of the points $H^+(W^+)$ and $H^-(W^-)$, located on the upper (y^+) and lower (y^-) crack edge respectively, when H (or W) approaches the tip R (or L) define the sign of the SIFs. This is a typical definition for the sign of K_I, which is positive when the crack edges open near the tip, that is when the y component of the point $H^+(W^+)$ displacement is higher than the correspondent component of point $H^-(W^-)$. Similarly, K_{II} at the tip R(L) is assumed positive if the x_R (x_L) displacement component of point $H^+(W^+)$ is higher than the corresponding component of the point $H^-(W^-)$.

The length to depth ratio $r = a/h$ was assumed as the (only) dimensionless parameter defining the geometry.

3. Mathematical formulation of the weight function

By considering the energy associated to a crack under linear elastic fracture mechanics hypothesis, Bueckner [16] and Rice [17] showed that the WF for a symmetrical one-tip crack can be obtained from the solution for one reference load condition. For a non-symmetrical problem, one reference load system is not enough as a mixed mode (I + II) of fracture is expected and the energy associated to the crack depends on both K_I and K_{II}. As discussed in [15,18,19], a matrix formulation of the WF is necessary to account the coupling effect, moreover, for an embedded crack the WF must provide the SIFs for both the tips.

In the examined problem, for which the geometry is symmetric about the y axis, any kind of loading can be considered as the superposition of a symmetric (S) and an anti-symmetric (A) components (see Fig. 2). After separating the load in symmetric and anti-symmetric components, the SIFs can be obtained by simple superposition. As a consequence, the WF domain can be restricted to one half of the crack length $[0, a]$ for both the tips in the related reference local system.

In the following several quantities are introduced in order to distinguish: crack loading mode (I or II), kind of component of the nominal stress applied on the crack faces (normal or tangential: σ, τ), type of loading (symmetric or anti-symmetric) and crack tip (R or L). In order to simplify the notation a sequence of four subscripts are used as defined in Table I.

The expressions (1) were adopted for the WF where, in order to simplify the notation, it was assumed $x = x_T$ i.e. $x = x_R$ for the tip R and $x = x_L$ for the tip L. This notation are applied also in the following:

$$\begin{pmatrix} K_{IR}(a,r) \\ K_{IIR}(a,r) \end{pmatrix} = \int_0^a \left[\begin{pmatrix} h_{I\sigma S}(x,a,r) & h_{I\tau S}(x,a,r) \\ h_{II\sigma S}(x,a,r) & h_{II\tau S}(x,a,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma_S(x) \\ \tau_S(x) \end{pmatrix} + \begin{pmatrix} h_{I\sigma A}(x,a,r) & h_{I\tau A}(x,a,r) \\ h_{II\sigma A}(x,a,r) & h_{II\tau A}(x,a,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma_A(x) \\ \tau_A(x) \end{pmatrix} \right] \cdot dx \tag{1a}$$

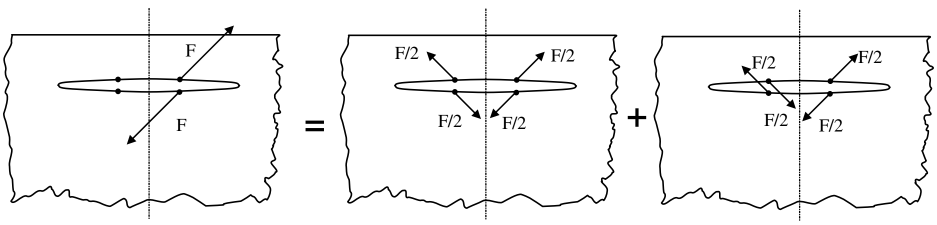

Fig. 2. A general loading condition expressed as a superposition of symmetric and anti-symmetric components.

Table I
Subscripts definition

Property	Dummy symbol	Occurrence
Mode of Fracture	M	I (opening mode), II (sliding mode)
Nominal stress component	μ	σ (normal stress), τ (shear stress)
Symmetry of the load	C	S (symmetric), A (anti-symmetric)
Crack tip	T	R (right), L (left)

$$\begin{pmatrix} K_{\mathrm{IL}}(a,r) \\ K_{\mathrm{IIL}}(a,r) \end{pmatrix} = \int_0^a \left[\begin{pmatrix} h_{\mathrm{I}\sigma\mathrm{S}}(x,a,r) & h_{\mathrm{I}\tau\mathrm{S}}(x,a,r) \\ h_{\mathrm{II}\sigma\mathrm{S}}(x,a,r) & h_{\mathrm{II}\tau\mathrm{S}}(x,a,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma_{\mathrm{S}}(x) \\ \tau_{\mathrm{S}}(x) \end{pmatrix} - \begin{pmatrix} h_{\mathrm{I}\sigma\mathrm{A}}(x,a,r) & h_{\mathrm{I}\tau\mathrm{A}}(x,a,r) \\ h_{\mathrm{II}\sigma\mathrm{A}}(x,a,r) & h_{\mathrm{II}\tau\mathrm{A}}(x,a,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma_{\mathrm{A}}(x) \\ \tau_{\mathrm{A}}(x) \end{pmatrix} \right] \cdot \mathrm{d}x \tag{1b}$$

It is worth noting that for the tip R(L) the anti-symmetrical terms have to be summed to (subtracted from) the symmetrical components.

The analytical expressions of the WF components were chosen in order to fulfil the asymptotical properties, In particular, as the r ratio approaches zero, the crack tends to behave as a Griffith crack in an infinite body for which the coupling effects vanish and the following relationships hold:

$$\begin{aligned} h_{\mathrm{I}\sigma\mathrm{S}}(x,a,r) &= h_{\mathrm{II}\tau\mathrm{A}}(x,a,r) = \frac{2}{\sqrt{\pi \cdot a}} \cdot \left(1 - \left(\frac{x}{a}\right)^2\right)^{-1/2} \\ h_{\mathrm{I}\sigma\mathrm{A}}(x,a,r) &= h_{\mathrm{II}\tau\mathrm{S}}(x,a,r) = \frac{2}{\sqrt{\pi \cdot a}} \frac{x}{a} \cdot \left(1 - \left(\frac{x}{a}\right)^2\right)^{-1/2} \\ h_{\mathrm{II}\sigma\mathrm{S}}(x,a,r) &= h_{\mathrm{I}\tau\mathrm{S}}(x,a,r) = h_{\mathrm{II}\sigma\mathrm{A}}(x,a,r) = h_{\mathrm{I}\tau\mathrm{A}}(x,a,r) = 0 \end{aligned} \tag{2}$$

Similar relationships hold for any values of r when $x_{\mathrm{T}} \to a$, as the WF has uncoupled universal properties at the crack tip.

When SIF solutions for reference conditions (left side of Eq. (1)) are known, the relationships (1) can be considered as integral equations having the WF components as unknowns. In order to solve the integral equations, the WF components for both the tips were expressed as (for subscript notation refer to Table I):

$$h_{\mathrm{M}\mu\mathrm{C}}(x,a,r) = \frac{2}{\sqrt{\pi \cdot a}} \cdot \sum_{i=0}^{n} \Gamma_{\mathrm{M}\mu\mathrm{C}i}(r) \cdot \left[1 - \left(\frac{x}{a}\right)^2\right]^{i-\frac{1}{2}} \tag{3}$$

for $h_{\mathrm{I}\sigma\mathrm{S}}(x,a,r)$, $h_{\mathrm{II}\sigma\mathrm{S}}(x,a,r)$, $h_{\mathrm{I}\tau\mathrm{A}}(x,a,r)$ and $h_{\mathrm{II}\tau\mathrm{A}}(x,a,r)$, and

$$h_{\mathrm{M}\mu\mathrm{C}}(x,a,r) = \frac{2}{\sqrt{\pi \cdot a}} \cdot \frac{x}{a} \cdot \sum_{i=0}^{n} \Gamma_{\mathrm{M}\mu\mathrm{C}i}(r) \cdot \left[1 - \left(\frac{x}{a}\right)^2\right]^{i-\frac{1}{2}} \tag{4}$$

for $h_{\mathrm{I}\sigma\mathrm{A}}(x,a,r)$, $h_{\mathrm{II}\sigma\mathrm{A}}(x,a,r)$, $h_{\mathrm{I}\tau\mathrm{S}}(x,a,r)$ and $h_{\mathrm{II}\tau\mathrm{S}}(x,a,r)$. As anticipated, for simplicity sake, it was indicated $x = x_{\mathrm{T}}$ in both equations.

It can be observed that the expansion (3) and (4) are even functions of x for symmetrical loading and odd functions for asymmetrical loading.

In order to fulfil the asymptotic conditions for $x_{\mathrm{T}} \to a$, the functions $\Gamma_{\mathrm{M}\mu\mathrm{C}0}(r)$ (i.e. for $i = 0$) of the diagonal and off-diagonal WF components are respectively:

$$\Gamma_{\mathrm{M}\mu\mathrm{S}0}(r) = \Gamma_{\mathrm{M}\mu\mathrm{A}0}(r) = 1 \quad \text{for } \mathrm{M}\mu = \mathrm{I}\sigma \text{ or } \mathrm{II}\tau \tag{5a}$$

$$\Gamma_{\mathrm{M}\mu\mathrm{S}0}(r) = \Gamma_{\mathrm{M}\mu\mathrm{A}0}(r) = 0 \quad \text{for } \mathrm{M}\mu = \mathrm{II}\sigma \text{ or } \mathrm{I}\tau \tag{5b}$$

In order to reproduce the dependence of the other $\Gamma(r)$ functions (with $i = 1 \ldots n$) on the r ratio, the following expression was found the be suitable in a wide range of r:

$$\Gamma_{\mathrm{M}\mu\mathrm{C}i}(r) = \frac{r^{\alpha_{\mathrm{M}\mu\mathrm{C}i}}}{(\beta_{\mathrm{M}\mu\mathrm{C}i})^{\alpha_{\mathrm{M}\mu\mathrm{C}i}} + r^{\alpha_{\mathrm{M}\mu\mathrm{C}i}}} \cdot \chi_{\mathrm{M}\mu\mathrm{C}i} \cdot (r)^{\delta_{\mathrm{M}\mu\mathrm{C}i}} + \varepsilon_{\mathrm{M}\mu\mathrm{C}i} \cdot (r)^{\phi_{\mathrm{M}\mu\mathrm{C}i}} \tag{6}$$

where the constant quantities $\alpha_{\mathrm{M}\mu\mathrm{C}i}$, $\beta_{\mathrm{M}\mu\mathrm{C}i}$, $\chi_{\mathrm{M}\mu\mathrm{C}i}$, $\delta_{\mathrm{M}\mu\mathrm{C}i}$, $\varepsilon_{\mathrm{M}\mu\mathrm{C}i}$, $\phi_{\mathrm{M}\mu\mathrm{C}i}$ were determined by least square fitting the SIF values calculated for reference loading conditions at several r ratios.

In order to limit the number of constants, the number of terms of the expansions (3) and (4) was limited to 3 ($n = 2$). The number of terms could be increased but, as demonstrated in the following, the accuracy of the obtained results indicated that the adopted approximation is adequate for practical applications. Numerical results are reported in Table II.

Table II
WF coefficients

Subscript	α	β	χ	δ	ε	ϕ
Symmetrical components						
IσS1	1.0329	2.1269	−0.91144	1.7134	1.1303	1.6909
IσS2	−1.7199	11.4784	−0.1988	1.2798	0.4190	1.5564
IτS1	1.5454	1.39166	−0.71464	0.69075	0.03522	1.10794
IτS2	2.6194	2.1332	0.2906	0.96017	−0.05087	1.2645
IIσS1	1.7356	3.5218	−0.04137	1.6725	−0.2071	1.4280
IIσS2	2.3517	1.0382	−0.4391	1.5019	0	5.9499
IIτS1	1.2876	2.4973	1.0930	0.7873	−0.12343	1.0637
IIτS2	2.1603	3.1472	−0.3035	1.2442	0.12322	1.40591
Anti-symmetrical components						
IσA1	1.7334	1.7244	1.14195	1.0485	0.06243	1.7198
IσA2	6.0063	0.90865	0.28073	1.81737	−0.03057	2.1718
IτA1	2.24873	1.84791	−0.26832	1.27526	0.17299	1.36238
IτA2	−1.98413	1.62476	0.29281	2.70492	−0.01265	1.39793
IIσA1	−1.66351	1.51638	0.51833	1.8608	−0.19715	1.52039
IIσA2	1.9993	1.2884	−0.4991	1.7098	0.1815	1.8354
IIτA1	1.2800	4.87025	1.07531	0.60087	0.01192	1.19241
IIτA2	1.62373	2.76249	−0.86390	0.80863	0.16463	0.99487

The reference solutions were obtained by the FE method [13,14] with a model developed in order to provide SIFs values with accuracy in the order of a few tenths of a percent. The quality of the model was assessed by analysing the condition of $r = 0$ (Griffith crack) for which the theoretical solution is known.

The evaluation of the coefficients in Eq. (6) needs at least two linearly independent reference loading conditions for any r ratio. Constant and linearly variable distributions of reference loading conditions were considered (as shown in Figs. 3 and 4) by applying appropriate tractions (either normal or shear) on the crack faces. Under this loading conditions symmetric and anti-symmetric WF components could be distinguished. A data base of SIFs values was built up by performing several FE analyses with different r ratios ranging from

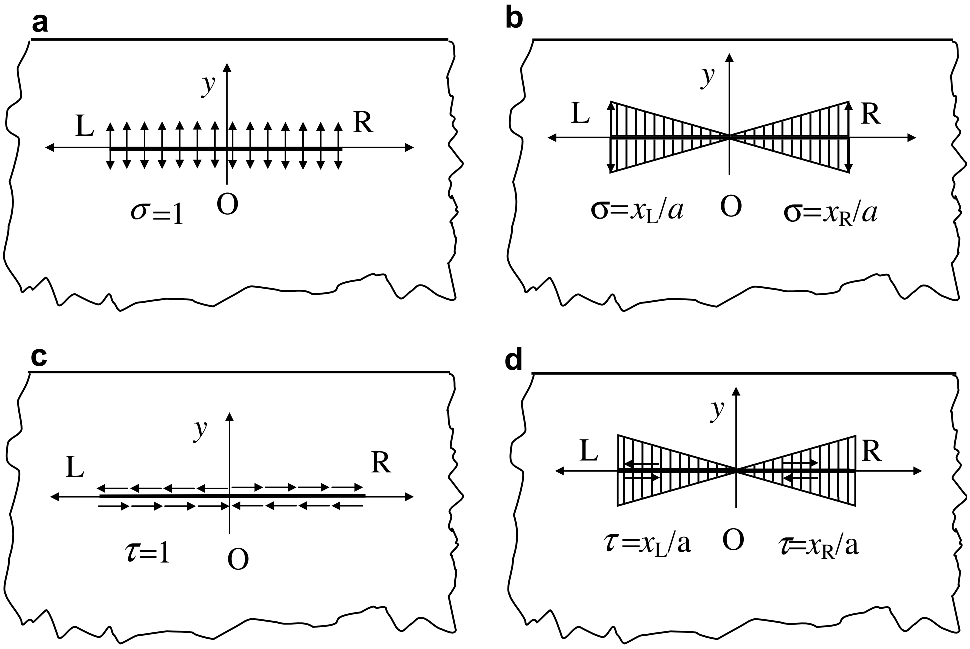

Fig. 3. Loading conditions used in the analysis for determining the symmetrical terms of the WF.

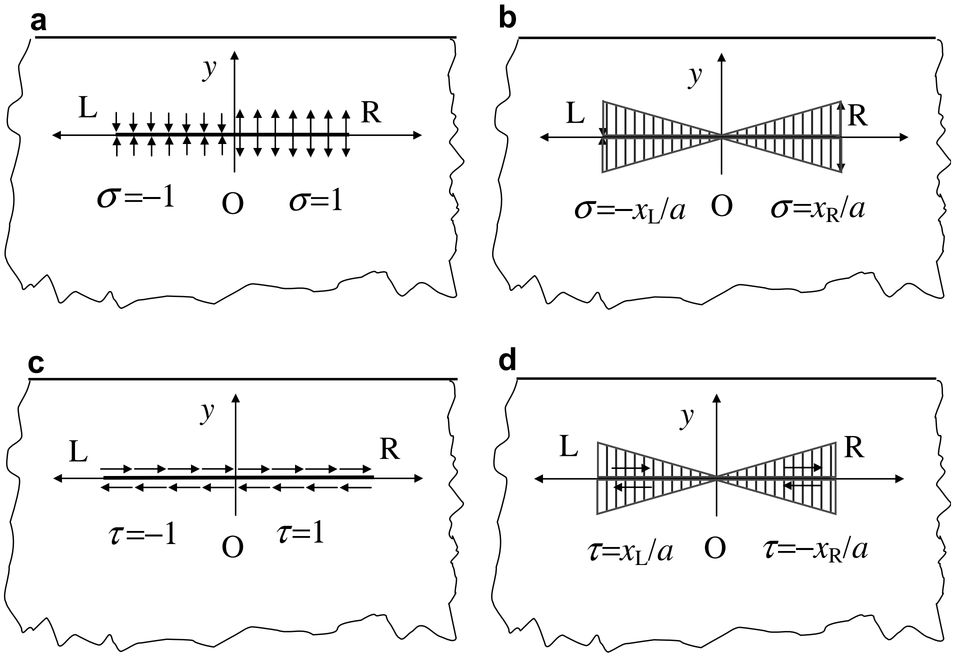

Fig. 4. Loading conditions used in the analysis for determining the anti-symmetrical terms of the WF.

$r = a/h = 0.005$ to include a defect so deep to be practically indistinguishable form a Griffith crack and $r = 40$ reproducing a shallow delamination immediately beneath the surface.

The SIFs calculated by the WF for the reference loading conditions were compared with the reference values obtained by direct FE analysis. The relative differences for both Modes generally were within 1.0% in the whole range of r. As these differences are comparable with the estimated level of accuracy for the direct FE SIF evaluation, the functions (6) assumed for representing the dependence to the r ratio were considered satisfactory.

4. Applicative examples

4.1. SIFs produced by uniform tractions on the crack faces

The SIFs produced by uniform normal traction (see Fig. 3a) and uniform shear traction (see Fig. 4c) are considered as examples to highlight the typical behaviour of the subsurface cracks as a function of r.

In Fig. 5 the values of K_{I} produced by the symmetric normal traction distribution and the K_{II} produced by an anti-symmetric shear traction are plotted as a function of r. The corresponding SIFs values of the Griffith crack were assumed as normalizing factors. The asymptotic properties of the crack as r approaches zero can be observed. The relative difference between the effective SIFs and to asymptotic values is lower than 4% for $r < 0.5$, i.e. for $h > 4a$, thus giving an estimate of the approximation level obtained by assuming the subsurface crack as a Griffith crack.

In Fig. 6 both the SIFs produced by the symmetric normal traction are plotted in order to show the coupling effect arising when the crack approaches the surface. For this loading condition, in which the Griffith analysis predict only mode I, K_{II} is nearly zero only for cracks far from the free surface, whereas when the crack approaches the surface, the absolute value of K_{II} becomes comparable with K_{I}. This coupling effect is a consequence of the loss of symmetry which is effective even as regard the x axis. Indeed, in this case, K_{II} is produced by the off-diagonal $h_{\mathrm{II}\sigma\mathrm{S}}(x,a,r)$ WF component. Similar results were obtained for any loading conditions.

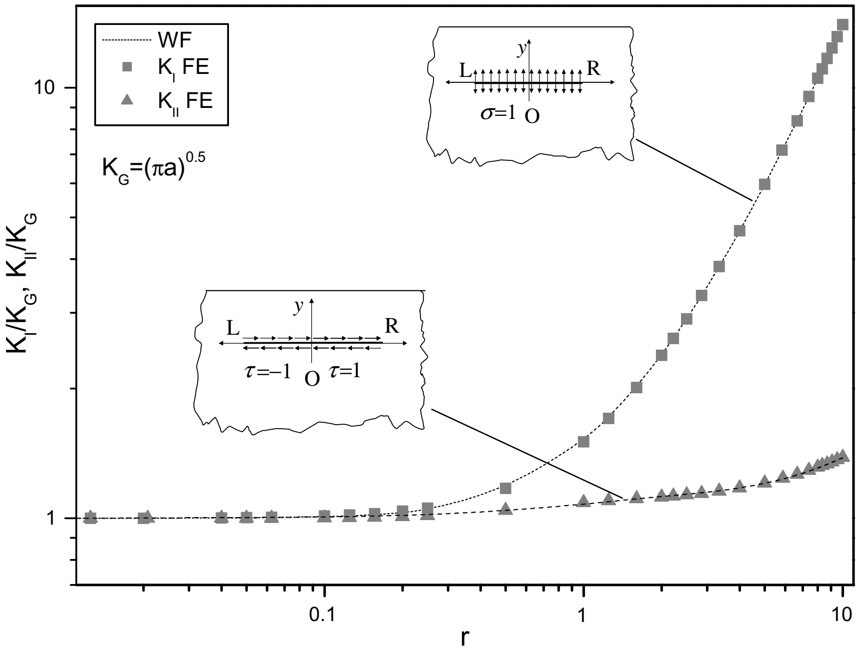

Fig. 5. Typical curves of SIFs versus r showing the effect of the free surface.

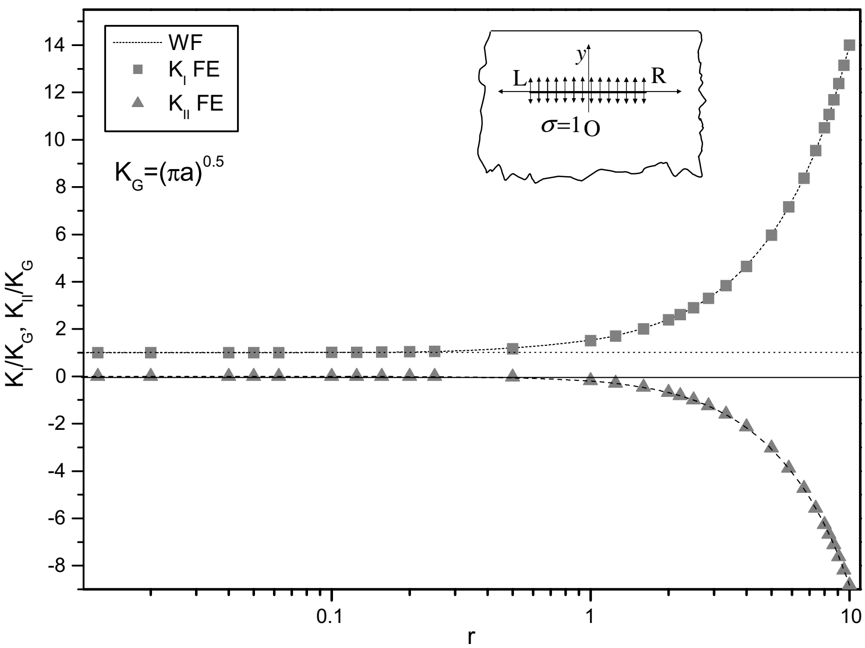

Fig. 6. Coupling effects as a function of r obtained for uniform normal traction applied on crack faces.

4.2. SIFs produced by a point like load travelling on the free surface

With reference to Fig. 7, a plane body carrying a subsurface crack loaded by a force having intensity P (force per unit thickness) is considered. The normal (P_n) and tangential (P_t) components were applied at the position represented by the algebraic quantity d (the abscissa in the x_R system). This is a simplest model

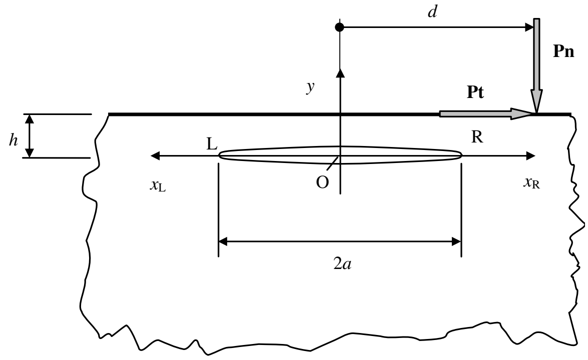

Fig. 7. Point like load moving on the surface.

for a travelling point-like contact, such that produced in a bearing or in a rail. Tangential component are produced by friction and inertia forces are neglected. It is worth noting that more accurate models of the contact load could be analysed by the proposed method by introducing more loading parameters in the analysis. Indeed, by the superposition principle, any complex pressure distribution can be approximated by a proper sequence of point like forces. In the present paper, this analysis was not reported in order to limit the number of parameters.

In the preliminary solution, no contact between crack edges was taken into account, and consequently material overlapping and negative K_I values are permitted, even though without physical meaning.

The analytical Boussinesq solution [20] for the stress produced either by P_n or P_t in the uncracked elastic semi-plane provided the basis for the evaluation of the nominal stress (normal and shear) components. In

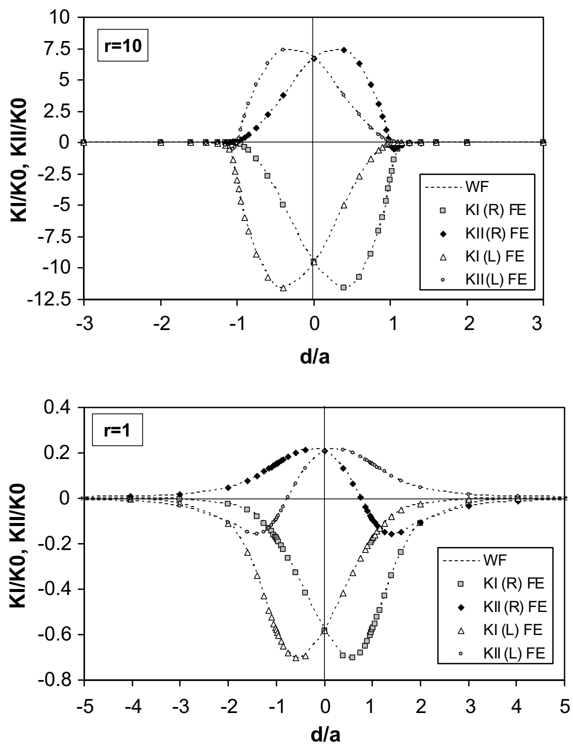

Fig. 8. SIF produced by the normal (compressive) force P_n travelling on the surface for two r ratios.

order to apply Eqs. (1), the nominal stress distributions were separated into symmetrical and anti-symmetrical components.

Several calculations were performed changing load positions (d/a) and r ratio for both force components P_n and P_t, and some results are reported in Figs. 8 and 9 together with the solutions of a complete FE analysis of the cracked semi-plane. SIF values are normalized with the characteristic value: $K_O = P \cdot \sqrt{\frac{\pi}{a}}$. In any analysed condition, the relative difference between FE and WF SIFs did not exceed 1%. As for this kind of load the nominal stress components are rather complex functions of the position (particularly for a cracks near the surface) the obtained agreement was considered a confirmation of the adequacy of the adopted expressions for the WF (4) and, particularly, of the number of terms in the truncated expansions (3) and (4).

When only the normal component of the load is applied, SIFs histories with K_I always negative are obtained (Fig. 8) for any load position and r ratio. Therefore, for this condition it is reasonable to predict a completely closed crack, subjected only to varying K_{II} during the load cycle produced by the movement of the force on the surface. On the contrary, when only tangential force is considered, complex K_I and K_{II} histories are obtained (Fig. 9) and conditions of partial or complete crack closure expected.

The effect produced by an inclined force travelling on the surface can be also evaluated, when neglecting the contact between crack surfaces, by simply superimposing the SIFs produced by the normal and tangential force components. The superposition is however consistent, from a physical point of view, only if the crack is completely open. Indeed, in the case of crack closure the evaluation of the SIFs becomes a non-linear problem, as the boundary conditions are unknown a priori and they depend on the applied load.

The knowledge of the closed crack region is necessary for evaluating the contact stresses mutually acting on the crack faces. If the contact stress is added to the nominal stress, the effective SIFs can be correctly determined also by the WF. The problem can be solved if the Green Function (GF) is known by which the COD components can be directly calculated by integration of the nominal plus contact stress on the crack faces.

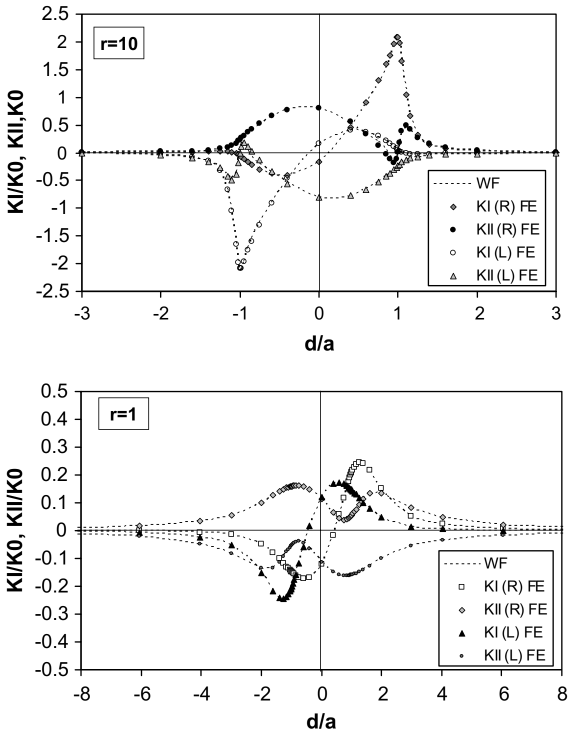

Fig. 9. SIF produced by the tangential force P_t travelling on the surface for two r ratios.

5. Analytical green's function and cod evaluation

The COD components u and v, indicating the relative displacements of corresponding points H^+, H^- on crack edges (Fig. 1), in the x_T and y direction respectively, can be calculated by using the WF. Similarly to the SIFs, also the COD components under any load can be calculated by adding symmetric and anti-symmetric COD components. By generalizing the formulation reported in [19] the following equations hold:

$$v_R(x,a,r) = \frac{2}{E'}\int_x^a [h_{I\sigma S}(x,b,r)\cdot K_{IS}(b,r) + h_{II\sigma S}(x,b,r)\cdot K_{IIS}(b,r)]\mathrm{d}b + \frac{2}{E'}\int_x^a [h_{I\sigma A}(x,b,r)\cdot K_{IA}(b,r) + h_{II\sigma A}(x,b,r)\cdot K_{IIA}(b,r)]\mathrm{d}b \quad (7a)$$

$$v_L(x,a,r) = \frac{2}{E'}\int_x^a [h_{I\sigma S}(x,b,r)\cdot K_{IS}(b,r) + h_{II\sigma S}(x,b,r)\cdot K_{IIS}(b,r)]\mathrm{d}b - \frac{2}{E'}\int_x^a [h_{I\sigma A}(x,b,r)\cdot K_{IA}(b,r) + h_{II\sigma A}(x,b,r)\cdot K_{IIA}(b,r)]\mathrm{d}b \quad (7b)$$

$$u_R(x,a,r) = \frac{2}{E'}\int_x^a [h_{I\tau S}(x,b,r)\cdot K_{IS}(b,r) + h_{II\tau S}(x,b,r)\cdot K_{IIS}(b,r)]\mathrm{d}b + \frac{2}{E'}\int_x^a [h_{I\tau A}(x,b,r)\cdot K_{IA}(b,r) + h_{II\tau A}(x,b,r)\cdot K_{II_A}(b,r)]\mathrm{d}b \quad (7c)$$

$$u_L(x,a,r) = \frac{2}{E'}\int_x^a [h_{I\tau S}(x,b,r)\cdot K_{IS}(b,r) + h_{II\tau S}(x,b,r)\cdot K_{IIS}(b,r)]\mathrm{d}b - \frac{2}{E'}\int_x^a [h_{I\tau A}(x,b,r)\cdot K_{IA}(b,r) + h_{II\tau A}(x,b,r)\cdot K_{IIA}(b,r)]\mathrm{d}b \quad (7d)$$

where E' is equal to E (Young modulus) for plane stress and $E/(1-\nu^2)$ for plane strain (ν is the Poisson's ratio). The functions $K_{MC}(b,r)$ are the SIFs produced by the nominal stress components in a fictitious sub-crack having half-length ($b \leqslant a$), that can be evaluated by Eq. (1) as follows:

$$\begin{pmatrix} K_{IC}(b,r) \\ K_{IIC}(b,r) \end{pmatrix} = \int_0^b \begin{pmatrix} h_{I\sigma C}(x',b,r) & h_{I\tau C}(x',b,r) \\ h_{II\sigma C}(x',b,r) & h_{II\tau C}(x',b,r) \end{pmatrix} \cdot \begin{pmatrix} \sigma_C(x') \\ \tau_C(x') \end{pmatrix} \cdot \mathrm{d}x' \quad (8)$$

with $x' = x_T$. It is worth noting that the sub-crack is located at the fixed depth h and, as a consequence, the dimensionless parameter r is equal to h/a. By indicating the WF in matrix notation:

$$[W_C(x,b,r)] = \begin{bmatrix} h_{I\sigma C}(x,b,r) & h_{I\tau C}(x,b,r) \\ h_{II\sigma C}(x,b,r) & h_{II\tau C}(x,b,r) \end{bmatrix} \quad (9)$$

considering Eqs. (7) and (8), after changing the order of integration [14], the following expression can be obtained:

$$\begin{bmatrix} v_R(x,a,r) \\ u_R(x,a,r) \end{bmatrix} = \frac{2}{E'}\cdot\int_0^a \left[\int_{\max(x,x')}^a [W_S(x,b,r)]^T \cdot [W_S(x',b,r)]\mathrm{d}b\right] \cdot \begin{pmatrix} \sigma_S(x') \\ \tau_S(x') \end{pmatrix} \cdot \mathrm{d}x' + \frac{2}{E'}\cdot\int_0^a \left[\int_{\max(x,x')}^a [W_A(x,b,r)]^T \cdot [W_A(x',b,r)]\mathrm{d}b\right] \cdot \begin{pmatrix} \sigma_A(x') \\ \tau_A(x') \end{pmatrix} \cdot \mathrm{d}x' \quad (10a)$$

$$\begin{bmatrix} v_L(x,a,r) \\ u_L(x,a,r) \end{bmatrix} = \frac{2}{E'}\cdot\int_0^a \left[\int_{\max(x,x')}^a [W_S(x,b,r)]^T \cdot [W_S(x',b,r)]\mathrm{d}b\right] \cdot \begin{pmatrix} \sigma_S(x') \\ \tau_S(x') \end{pmatrix} \cdot \mathrm{d}x' - \frac{2}{E'}\cdot\int_0^a \left[\int_{\max(x,x')}^a [W_A(x,b,r)]^T \cdot [W_A(x',b,r)]\mathrm{d}b\right] \cdot \begin{pmatrix} \sigma_A(x') \\ \tau_A(x') \end{pmatrix} \cdot \mathrm{d}x' \quad (10b)$$

where $[\,]^T$ is the transpose matrix. By introducing the following 2×2 matrices:

$$G_S(x,x',r) = \begin{bmatrix} G_{v\sigma S}(x,x',r) & G_{v\tau S}(x,x',r) \\ G_{u\sigma S}(x,x',r) & G_{u\tau S}(x,x',r) \end{bmatrix} = \int_{\max(x,x')}^{a} [W_S(x,b,r)]^T \cdot [W_S(x',b,r)] db \tag{11a}$$

$$G_A(x,x',r) = \begin{bmatrix} G_{v\sigma A}(x,x',r) & G_{v\tau A}(x,x',r) \\ G_{u\sigma A}(x,x',r) & G_{u\tau A}(x,x',r) \end{bmatrix} = \int_{\max(x,x')}^{a} [W_A(x,b,r)]^T \cdot [W_A(x',b,r)] db \tag{11b}$$

Eq. (10) can be rewritten as:

$$\begin{bmatrix} v_R(x,a,r) \\ u_R(x,a,r) \end{bmatrix} = \frac{2}{E'} \cdot \left\{ \int_0^a [G_S(x,x',r)] \cdot \begin{pmatrix} \sigma_S(x') \\ \tau_S(x') \end{pmatrix} dx' + \int_0^a [G_A(x,x',r)] \cdot \begin{pmatrix} \sigma_A(x') \\ \tau_A(x') \end{pmatrix} \cdot dx' \right\} \tag{12a}$$

$$\begin{bmatrix} v_L(x,a,r) \\ u_L(x,a,r) \end{bmatrix} = \frac{2}{E'} \cdot \left\{ \int_0^a [G_S(x,x',r)] \cdot \begin{pmatrix} \sigma_S(x') \\ \tau_S(x') \end{pmatrix} dx' - \int_0^a [G_A(x,x',r)] \cdot \begin{pmatrix} \sigma_A(x') \\ \tau_A(x') \end{pmatrix} \cdot dx' \right\} \tag{12b}$$

which demonstrates that $[G_C(x,x',r)]$ are the Green's functions (GF) as they relate the load applied on the crack faces to the displacement of the points of load application.

Considering the power law expansions (3) and (4) adopted for the WF and taking into account of the combination between the WF components in the matrix product (11), the GF can be obtained analytically by solving $3 \cdot (n+1)^2$ integrals of one variable. With $n = 2$ the number of integrals is 27, however, considering the not zero terms after applying the asymptotic properties, only 22 integrals have to be calculated. By adopting a recursive procedure, the solution can be reduced to the following three classes of integrals where $k,j = 0\ldots n$:

$$I_{1kj}(x,x') = \int_{\max(x,x')}^{a} \frac{1}{b} \cdot \left(1 - \frac{x}{b}\right)^{\left(k-\frac{1}{2}\right)} \cdot \left(1 - \frac{x'}{b}\right)^{\left(j-\frac{1}{2}\right)} db \tag{13a}$$

$$I_{2kj}(x,x') = \int_{\max(x,x')}^{a} \frac{1}{b^2} \cdot \left(1 - \frac{x}{b}\right)^{\left(k-\frac{1}{2}\right)} \cdot \left(1 - \frac{x'}{b}\right)^{\left(j-\frac{1}{2}\right)} db \tag{13b}$$

$$I_{3kj}(x,x') = \int_{\max(x,x')}^{a} \frac{1}{b^3} \cdot \left(1 - \frac{x}{b}\right)^{\left(k-\frac{1}{2}\right)} \cdot \left(1 - \frac{x'}{b}\right)^{\left(j-\frac{1}{2}\right)} db \tag{13c}$$

Integrals I_1 and I_3 can be expressed as combinations of elementary functions whereas I_2 can be reduced to elliptic integrals.

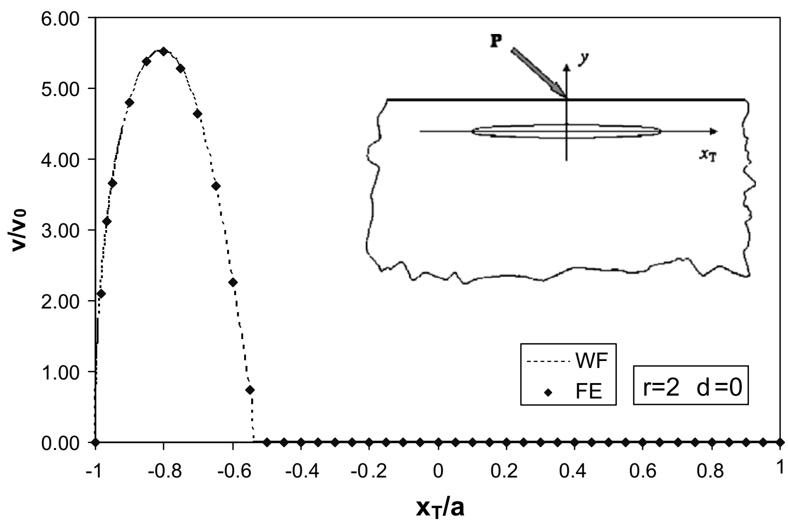

Fig. 10. COD v component calculated by the GF compared to the non-linear FE solution. COD are normalized by the characteristic parameter $v_0 = 0.02P/\pi E$.

By knowing the GFs, Eq. (12) can be used to obtain the COD components at any location of the crack for any load condition when the nominal stress components $\sigma(x)$ and $\tau(x)$ are known on the crack edge. By extending the procedure explained in [19] to the case of a two-tip crack, the problem of crack closure can be faced in an efficient way too. Indeed, in the region where the crack faces are in contact the mutual contact traction (unknown) has to be added to the nominal stress, but, in this region the COD is known. This condition can be used to determine the contact stress by adopting an iterative procedure for determining the location and extension of the contact region.

An example of complete COD evaluation by means of the GF under crack closure, is shown in Fig. 10. It can be observed that, also in this case, the results are in good agreement with the solutions of a non-linear FE analysis including the contact on the crack faces. It is interesting to observe the an external force inclined by an angle of 45° produces partial closure with the left tip open. This implies that in some conditions the combined effect of nominal stresses and contact tractions produces on the left tip a positive K_I even though the K_I calculated by linear superposition (when neglecting the contact between the crack faces) is negative.

6. Conclusion

An analytical formulation of the WF for a crack parallel to the surface of a semi-infinite elastic plane was proposed. The results of a Finite Element analysis carried out for several independent loading cases were used for fitting the coefficients of a truncated power expansion. The WF reproduced the FE results with an accuracy in the order of 1% for a wide range of length to depth ratios.

The effect of a point-like load travelling on the surface of a semi-infinite body with a subsurface crack was studied. The conditions of partial crack closure were initially neglected and, also under this kind of loading producing complex nominal stress distributions, the results were in very good agreement with those determined by FE analysis. In the presence of typical load applied on the surface, negative K_I values were usually obtained thus suggesting that the crack is partially or totally closed. This indicated that the correct crack analysis requires the evaluation of the COD components. The Green's Function (GF) was deduced by the WF in order to obtain an efficient method to evaluate the COD components and, consequently, the effects of the contact stresses on the effective SIFs.

References

[1] Glodez S, Ren Z, Flasker J. Surface fatigue of gear teeth flanks. Comput Struct 1999;73(1):475–83.
[2] Ringsberg JW. Life prediction of rolling contact fatigue crack initiation. Int J Fatigue 2001;23(7):575–86.
[3] Glodez S, Ren Z. Modelling of crack growth under cyclic contact loading. Theor Appl Fract Mech 1998;30:159–73.
[4] Ding Y, Jones R, Kuhnell B. Numerical analysis of subsurface crack failure beneath the pitch line of a gear tooth during engagement. Wear 1995;185(1):141–9.
[5] Lukas P, Kunz L. Specific features of high-cycle and ultra-high-cycle fatigue. Fatigue Fract Engng Mater Struct 2002;25(8–9):747–53.
[6] Pippan R, Tabering B, Gach E, Riemelmoser F. Non-propagation conditions for fatigue cracks and fatigue in the very high-cycle regime. Fatigue Fract Engng Mater Struct 2002;25(8-9):805–11.
[7] Wang QY, Bathias C, Kawagoishi N, Chen Q. Effect of inclusion on subsurface crack initiation and gigacycle fatigue strength. Int J Fatigue 2002;24(12):1269–74.
[8] Yang F, Qian C, Li JCM. Finite element analysis of a subsurface crack. Int J fract 1996;77:337–50.
[9] Kudish II, Burris KW. Modeling of surface and subsurface crack behaviour under contact load in the presence of lubricant. Int J fract 2004;125:125–47.
[10] Komvopoulos K, Cho S-S. Finite element analysis of subsurface crack propagation in a half-space due to a moving asperity contact. Wear 1997;209:57–68.
[11] Aslantas K, Tasgetiren S. Modelling of spall formation in a plate made of austempered ductile iron having a subsurface-edge crack. Comput Mater Sci 2004;29:29–36.
[12] Danielli P. Ricerca sulla progettazione, mediante criteri tribologici, di ingranaggi avanzati per impieghi aeronautici o aerospaziali, PhD thesis, University of Pisa, 2000, (In Italian Language).
[13] Beghini M, Bertini L, Fontanari V. Stress intensity factors for a Subsurface crack in a two dimensional half-space. In: Proceedings of IGF 2004, XVII Congress of the Italian Group of Fracture, 15–18 June, 2004, Bologna (Italy) (on CD-Rom).
[14] Beghini M, Bertini L, Fontanari V. A general Weight Function for a subsurface crack in a two dimensional half-space. In: Proceedings of ICF11 – International Congress of Fracture, Turin (Italy), 20–25 March, 2005, (on CD-ROM).
[15] Fett T, Muntz D. Stress intensity factors and weight functions. Comp Mech Pu 1997.

[16] Bueckner HF. A novel principle for the computation of stress intensity factors. Zeitung fuer Angewandte Mathematische Mechanik 1970;50:529–46.

[17] Rice JR. Some remarks on the elastic crack-tip stress fields. Int J Solids Struct 1972;8:751–8.

[18] Beghini M, Bertini L, Fontanari V. Weight function for an inclined edge crack in a semiplane. Int J Fract 1999;99:281–92.

[19] Beghini M, Bertini L, Fontanari V. A weight function technique for partially closed inclined edge crack analysis. Int J Fract 2001;112:57–68.

[20] Timoshenko SP, Goodier JN. Theory of elasticity. Mc Graw-Hill; 1970.