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Modeling of elastic transversely isotropic composite using the asymptotic homogenization method. Some comparisons with other models

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Abstract

The objective of this paper is to apply the asymptotic homogenization method (AHM) to determine the analytical formulae for the elastic effective coefficients of a two-phase fibrous composite provided with a periodic structure. In the analysis, the periodicity of the structure is assumed to be much smaller than the elastic wavelength. The fibres are aligned unidirectional with respect to the x_3-axis. The constituents are transversely isotropic materials. The results are used to determine numerically the linear elastic behavior of two types of fibre composites. Some comparisons with different experimental results and theoretical models are shown.

Keywords: Effective properties; Fibrous composite; Homogenization; Transversely isotropic; Periodic structure

1. Introduction

The asymptotic homogenization method (AHM), which was developed by Bensoussan et al. [1], Sanchez-Palencia [2] and Bakhvalov and Panasenko [3], is a mathematically rigorous technique for predicting both the local and global properties of this kind of inhomogeneous media. The main problem of the AHM is that averaged coefficients depend on the solutions of the so-called local problems in the periodic cell [4]. These problems are given by a set of partial differential equations with periodic boundary conditions and their solution, in general, requires numerical methods [5].

In the present work, an analytical expression and exact formulae for all effective elastic coefficients are obtained using the AHM for an unidirectional reinforced two-phase composite with transversely isotropic cylindrical fibres periodically distributed in a matrix [6–8]. In the analysis, the periodicity of the structure is assumed to be much smaller than the elastic wavelength. A comparison with different models [9–11],

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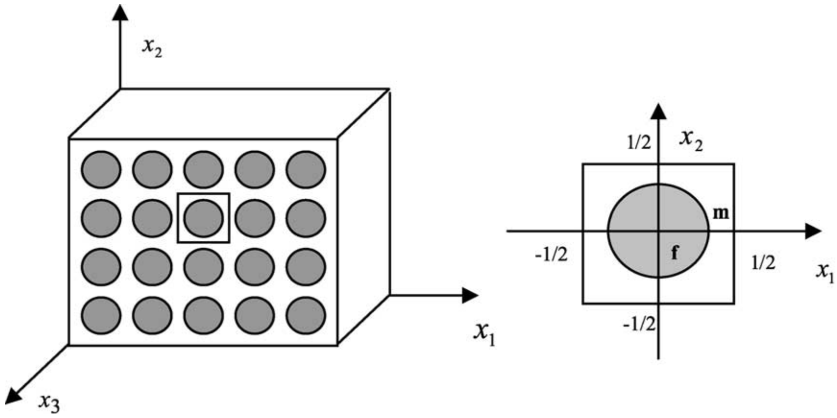

Fig. 1. Geometric distribution of the reinforcements in the composite and its periodic square cell.

and some experimental results [12,15,14] is presented. In Section 2, the AHM is briefly explained, the statement of the problem and the corresponding local problems are defined in order to obtain the overall properties of the elastic composite considered for the study. The general expressions of local problems and effective coefficients for elastic heterogeneous media with a periodic structure are summarized. In Section 3, the averaged formulae obtained are compared with other models and experimental results.

2. Formulation and statement of the local problems

The constitutive relations of the linear elasticity theory for an heterogeneous and periodic medium, Ω, is characterized by the Y-periodic function C. Y denotes the periodic cell, whereas C is the elastic fourth order tensor. By mean of AHM, the initial constitutive relations with rapidly oscillating material coefficients is transformed in new physical relations with constant coefficients $\overline{C}$, which represent the elastic properties of an equivalent homogeneous medium and are called the effective coefficients of Ω.

The main problem to obtain such average formulae is to find the Y-periodic solutions $U_{k(pq)}$ of the **local problems on** Y in terms of the fast variable $\boldsymbol{y}$ [7].

Once the **local problems** is solved, the homogenized moduli $\overline{C}_{ijpq}$ may be determined by using the following formulae:

$$\overline{C}_{ijpq} = \langle C_{ijpq} + C_{ijkl} U_{k(pq),l} \rangle. \tag{1}$$

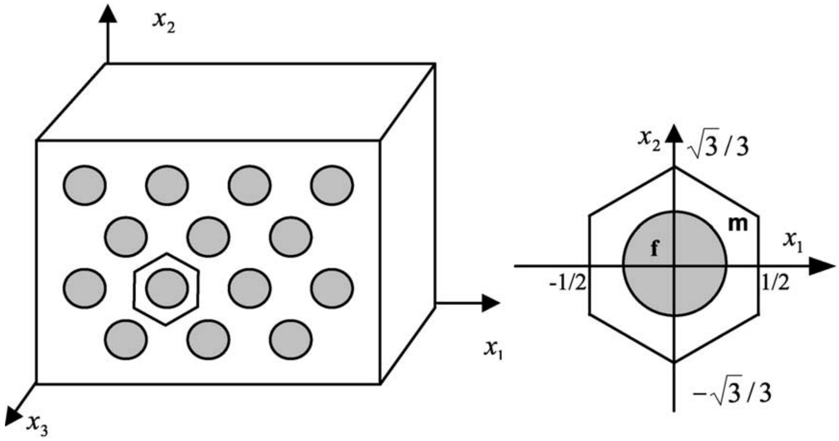

Fig. 2. Geometric distribution of the reinforcements in the composite and its periodic hexagonal cell.

where

$$\langle F\rangle = \frac{1}{|X|}\int_X F\,\mathrm{d}X.$$

The unit cell Y of the body is chosen with a side parallel to the x_3, with unit length in the x_3-direction. The transversal sections of the periodic cell are: (A) unit squares (Fig. 1), (B) regular hexagon (Fig. 2), both with the radius of the fibres denoted by r. Due to the periodic distributions of the fibres in the isotropy plane Ox_1x_2, it is possible to reduce the general problem to the solution of the local problems over the unit cell. In this case, the elasticity tensor components C_{ijkl} takes different values in the regions occupied by these two different materials, such that

$$C_{ijkl}(y_1,y_2) = \begin{cases} C^{(1)}_{ijkl} \text{ if } (y_1,y_2)\in M_1 \text{ (matrix)} \\ C^{(2)}_{ijkl} \text{ if } (y_1,y_2)\in M_2 \text{ (fibre)} \end{cases}.$$

The local problems can be written as

$$\tau^{(\alpha)}_{ij(pq)},j = 0 \text{ if } (y_1,y_2)\in M_\alpha, \tag{2}$$

with $\tau^{\alpha}_{ij(pq)} = C^{\alpha}_{ijkl}U^{(\alpha)}_{k(pq),l}$ (α, l, j = 1, 2, and i, k, p, q = 1, 2, 3).

The solution of problem (2) must consist of doubly periodic functions in y_1 and y_2 subject to the following perfect bounding conditions at the interface Γ:

$$U^{(1)}_{k(pq)}|_\Gamma = U^{(2)}_{k(pq)}|_\Gamma \tag{3}$$

$$(\tau^{(1)}_{ij(pq)} + C^{(1)}_{ijpq})n^{(1)}_j|_\Gamma = -(\tau^{(2)}_{ij(pq)} + C^{(2)}_{ijpq})n^{(2)}_j|_\Gamma. \tag{4}$$

The potential method of complex variables and the properties of doubly periodic Weierstrass and related functions are used for the solution of the local problems (Eqs. (2)–(4)). In that way, we obtain for the average coefficients of the composite given in Figs. 1 and 2 the following analytic and closed form formulae using the abbreviated notation of two indices:

$$\overline{C}_{11} = \overline{C}_{22} = \langle C_{11}\rangle + \lambda(\Delta_2^2\alpha_1/C^{(1)}_{66} - \Delta_1\alpha_2),$$

$$\overline{C}_{12} = \langle C_{12}\rangle + \lambda(\Delta_2^2\alpha_1/C^{(1)}_{66} - \Delta_1\alpha_2),$$

$$\overline{C}_{13} = \overline{C}_{23} = \langle C_{13}\rangle + \lambda\Delta_2\Delta_3\alpha_1/C^{(1)}_{66},$$

$$\overline{C}_{33} = \langle C_{13}\rangle + \lambda\Delta_3^2\alpha_1/C^{(1)}_{66},$$

$$\overline{C}_{66} = \langle C_{66}\rangle + \lambda\Delta_1\alpha_3, \text{ if } \mu = \pi/2,$$

$$\overline{C}_{66} = \frac{\overline{C}_{11} - \overline{C}_{12}}{2}, \text{if } \mu = \pi/3,$$

$$\overline{C}_{44} = \overline{C}_{55} = \langle C_{55}\rangle + \lambda\Delta_4\alpha_4, \tag{5}$$

where,

$$\Delta_1 = C^{(1)}_{66} - C^{(2)}_{66}, \quad \Delta_2 = C^{(1)}_{11} - C^{(2)}_{11} + C^{(1)}_{12} - C^{(2)}_{12},$$

$$\Delta_3 = C^{(1)}_{13} - C^{(2)}_{13}, \quad \Delta_4 = C^{(1)}_{55} - C^{(2)}_{55},$$

$$\alpha_1 = \frac{\kappa_2 - 1}{2\alpha_o}\left(\lambda - 1 - B(\kappa_1+1)\frac{\kappa_2 - 1}{2\alpha_o}\mathbf{N}_1 Z^{-1}\mathbf{N}_2\right),$$

$$\alpha_2 = \frac{\kappa_1 + 1}{(1+\chi^*\kappa_1)(\lambda_2 - B^2\mathbf{U}_1\mathbf{U}^{-1}\mathbf{U}_2)} - 1,$$

$$\alpha_3 = 1 - \frac{(\kappa_1+1)BC^{(1)}_{66}}{\lambda_3 - B^2\mathbf{V}_1\mathbf{V}^{-1}\mathbf{V}_2},$$

$$\alpha_4 = 1 - \frac{2C^{(1)}_{55}}{(1+\chi^*)(1+\chi\lambda - B^2\mathbf{N}_1 Y^{-1}\mathbf{N}_2)},$$

$$\lambda = \pi r^2/\sin\mu, \quad \mu = \frac{\pi}{2} \quad \text{or} \quad \mu = \frac{\pi}{3}. \tag{6}$$

The bar above the material constants means the effective coefficients of the composite and λ is the volume fraction of the fiber. The magnitudes involved in the expressions of α_1, α_2, α_3, α_4 in Eq. (6) can be found in detail in Refs. [6,7]. The most difficult aspect in Eq. (6) is the calculation of infinite numerical matrices $\mathbf{N}_1$, $\mathbf{Z}$, $\mathbf{N}_2$, $\mathbf{U}_1$, $\mathbf{U}$, $\mathbf{U}_2$, $\mathbf{V}_1$, $\mathbf{V}$, $\mathbf{V}_2$, and $\mathbf{Y}$, which can be computed according with the proposed algorithm given in Ref. [8].

3. Comparisons with different models

The overall properties calculated in Eq. (5) for the hexagonal and square cells of both types of composites (see Figs. 1 and 2) are compared with theoretical and experimental results reported in the above mentioned papers.

(1) In Ref. [9], theoretical expressions are obtained by means of classical theory of elasticity for determining the composite elastic constants for fibre reinforced plastics in terms of the elastic moduli and the geometrical parameters of the constituents. This investigations were made for a hexagonal array and the material properties used were E-glass fibre and epoxy resin. The fibre volume content was approximately 63%. A good concordance between the numerical results reported in Ref. [9] and the numerical calculations derived from Eq. (5) were obtained and they are shown in Table 1.

(2) Numerical results are obtained in Ref. [10] from simple explicit expressions. Several models for calculating elastic constants of fibre-reinforced composites with transversely isotropic constituents for hexagonal lattice model are discussed in this work. The material constituents data are graphite fibres embedded in an epoxy matrix. The fibre volume content is approximately 50%. The results of AHM and the predictions in the model [10] for the Young's moduli, shear modulus and Poisson's ratios are very close. They are shown in Table 2.

(3) The complete set of elastic mechanical properties for a composite reinforced by graphite in an epoxy matrix was investigated in Ref. [12]. In this work, equations used to calculate the complete set of elastic transversely isotropic properties for unidirectional fibre-reinforced materials having transversely isotropic fibres were experimentally verified by using improved ultrasonic techniques. In Table 3, the AHM formulae are compared with the results of Ref. [12] for a composite Modmor II/LY558 at 67% of volume fraction of the fibres.

(4) The Hill's relations [13] are satisfied identically by the effective coefficients Eq. (5). This work showed proof that for the calculated effective coefficients, these universal relations are constant and invariant in relation with the volume fraction $\frac{\Delta_2}{\Delta_3} = \frac{\bar{k}-\langle k\rangle}{\bar{C}_{13}-\langle C_{13}\rangle} = \frac{\bar{C}_{13}-\langle C_{13}\rangle}{\bar{C}_{33}-\langle C_{33}\rangle}$, where $k=0.5(C_{11}+C_{12})$.

(5) The elastic constants C_{ij} were measured and calculated for a laminated uniaxially fibre-reinforced boron–aluminum composite in Ref. [14]. In this paper, three theoretical models were considered: square array, hexagonal array and random-distribution and relationships for predicting the full set of elastic constants for this model were derived. A comparison of AHM given by Eq. (5) with the experimental and theoretical results reported in Ref. [14] at a volume fraction of 48% is shown in Table 4. The hexagonal configuration by AHM agrees best with random distribution model and the "observed". Considering all six elastic constants, "observed" and AHM differ on the average by 6% for hexagonal, and 15% for square.

(6) A comparison between the theoretical results derived here and the experimental data, which were obtained for transversely isotropic Modmor type 1

Table 1
Comparisons between AHM for both type of arrays (hexagonal and square) and the model reported in [9]

Models used	$\bar{C}_{11}$	$\bar{C}_{12}$	$\bar{C}_{13}$	$\bar{C}_{33}$	$\bar{C}_{44}$	$\bar{C}_{66}$
Chen and Cheng	2.2352	0.8933	0.7625	6.8621	0.6926	0.6727
(5) Hexagonal	2.2369	0.8919	0.7626	6.8631	0.6934	0.6725
(5) Square	2.5749	0.6855	0.7872	6.8723	0.7467	0.5245

Table 2
Theoretical results of AHM and the predictions given in Ref. [10] for the Young's and shear moduli and Poisson's ratios

	$\bar{E}_a$	$\bar{E}_t$	$\bar{G}_a$	$\bar{G}_t$	$\bar{\nu}_a$	$\bar{\nu}_t$
AHM	12.3	1.0654	0.6020	0.4037	0.3000	0.3195
Behrens	12.3	1.05805	0.6014	0.3994	0.3002	0.3205

Table 3
Comparison between the AHM formulae and the results reported by Kriz–Stinchcomb [12]

	$\bar{C}_{11}$	$\bar{C}_{12}$	$\bar{C}_{13}$	$\bar{C}_{33}$	$\bar{C}_{44}$	$\bar{C}_{66}$
AHM	14.5511	7.4083	6.6223	161.2077	7.2313	3.5714
Kriz–Stinchcomb	14.5	7.24	6.50	161	7.10	3.63

Table 4
Comparison of AHM with the experimental and theoretical results reported in Ref. [14] for both types of arrays

$\bar{C}_{ij}$	Observed	Square model	Hexagonal model	Random model	AHM hexagonal	AHM square
$\bar{C}_{11}$	1.852	1.856	1.872	1.790	1.7993	1.8802
$\bar{C}_{12}$	0.779	–	0.661	0.745	0.7336	0.6570
$\bar{C}_{13}$	0.606	–	0.578	0.583	0.5832	0.5832
$\bar{C}_{33}$	2.450	2.480	2.551	2.560	2.5601	2.5601
$\bar{C}_{44}$	0.566	0.451	0.561	0.559	0.5595	0.5634
$\bar{C}_{66}$	0.526	–	0.606	0.523	0.5328	0.4737

carbon fibres in isotropic Ciba LY558 epoxy resin by Dean now follows. Fig. 3a and b plots the effective stiffness $\bar{C}_{11}$ and $\bar{C}_{33}$ versus the volume fraction of the fibre λ. The solid line gives the results using the analytical formula (5). The empty circles are the experimental values. It can be seen that the experimental data agree quite well with the prediction of the asymptotic homogenization method. In Fig. 3c and d, the transverse shear modulus $\bar{G}_t$ and the effective Poisson's ratio $\bar{\nu}_t$ are plotted against the volume fraction λ. The predictions given by asymptotic homogenization are compared with the theoretical expressions bounds, derived by Ref. [11], which predict the composite properties as a function of the properties of the constituents and their concentration. The upper and lower bounds and the AHM prediction are very close. Their curves are almost indistinguishable. The solid lines of Fig. 3 have been calculated for

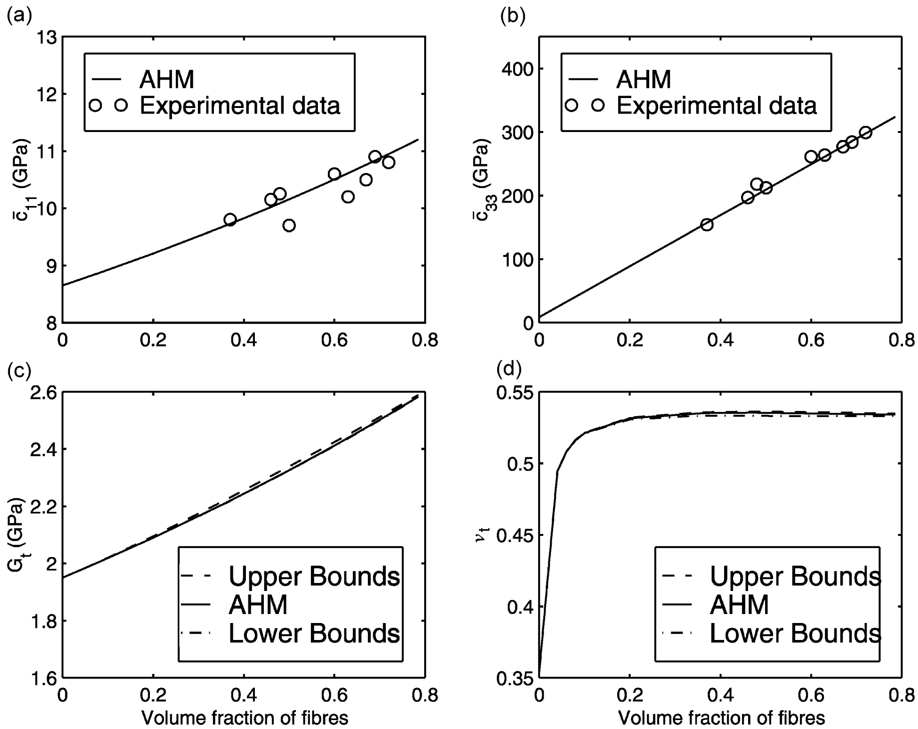

Fig. 3. (a–b) Comparison between predicted and the measured [14] values of the effective coefficients $\bar{C}_{11}$, $\bar{C}_{33}$ for transversely isotropic Modmor type 1 carbon/Ciba LY558 epoxy resin versus the volume fraction of fibers. (c–d) Comparison between the effective shear modulus $\bar{G}_t$ and the effective Poisson's ratio $\bar{\nu}_t$ with the Hashin–Rosen bounds.

the hexagonal distribution of the fibres. Similar results can be obtained for the square distribution.

4. Concluding remarks

The overall coefficients were calculated using the asymptotic homogenization technique. The comparisons in the present work between formula (5), other models and experimental results proved the effectiveness approach of the asymptotic homogenization for the derivation of the overall properties.

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