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On the K-divisibility constant for some special finite-dimensional Banach couples

Yacin Ameur [a,*], Michael Cwikel [b,1]

[a] Royal Institute of Technology, Mathematics Department, Lindstedtsvagen 25, SE-100 44 Stockholm, Sweden
[b] Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel

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ABSTRACT

We prove new estimates of the K-divisibility constants for some special Banach couples. In particular, we prove that the K-divisibility constant for a couple of the form $(U \oplus V, U)$ where U and V are non-trivial Hilbert spaces equals $2/\sqrt{3}$. We also prove estimates for the K-divisibility constant of the two-dimensional version of the couple (L_2, L_∞), proving in particular that this couple is not exactly K-divisible. There are also several auxiliary results, including some estimates for relative Calderón constants for finite-dimensional couples.

Contents

* Corresponding author.
E-mail addresses: yacin.ameur@gmail.com (Y. Ameur), mcwikel@math.technion.ac.il (M. Cwikel).
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1. Introduction

Let us begin by recalling the celebrated Brudnyi–Krugljak K-divisibility theorem (cf. [6], [7, p. 325, Paragraph C and Theorem 3.2.7]).

Theorem 1.1. *Let $\vec{A} = (A_0, A_1)$ be a Banach couple, and let N be either a fixed natural number or ∞. There exists a constant C_N, depending only on $\vec{A}$ and N, which has the following property: Suppose that a is an arbitrary element of $A_0 + A_1$ whose Peetre K-functional satisfies the estimate*

$$K(t, a; \vec{A}) \leqslant \sum_{n=1}^{N} \phi_n(t) \quad \text{for all } t > 0, \tag{1.1}$$

where the functions ϕ_n are each positive and concave on $(0, \infty)$ and $\sum_{n=1}^{N} \phi_n(1) < \infty$. Then there exists a sequence of elements $a_n \in A_0 + A_1$ such that $a = \sum_{n=1}^{N} a_n$ (where this series converges in $A_0 + A_1$ norm) and

$$K(t, a_n; \vec{A}) \leqslant C_N \phi_n(t) \quad \text{for all } t > 0 \text{ and each } 1 \leqslant n < N + 1. \tag{1.2}$$

The main interest of Theorem 1.1 resides in the special case when $N = \infty$, but we will also need to consider other values of N below. We refer to [7] and also to remarks in the introductions of [9] and [11] for more details about Theorem 1.1 and its applications. Its original proof appears in [7]. Various alternative proofs using the so-called "strong fundamental lemma" can be found in [9,4], cf. also [3].

We shall use the notation $\gamma_N(\vec{A})$ for the infimum of all numbers C_N having the property stated in Theorem 1.1. This number may be called the *N-term K-divisibility constant for $\vec{A}$*. When $N = \infty$, we follow the notation and terminology of previous papers and simply write $\gamma(\vec{A})$ instead of $\gamma_N(\vec{A})$ and speak of the *K-divisibility constant* of $\vec{A}$. It is not hard to check that these constants satisfy

$$1 \leqslant \gamma_i(\vec{A}) \leqslant \gamma_j(\vec{A}) \leqslant \gamma(\vec{A}), \quad 1 \leqslant i \leqslant j.$$

(Strictly speaking, the first inequality is only true if $\vec{A}$ is *non-zero*. "Non-zero" means that we exclude the trivial cases where $A_0 = A_1$ and this space consists solely of the zero element of some Hausdorff topological vector space. In these cases $\gamma(\vec{A}) = 0$.)

All Banach spaces in this paper will be assumed to be over the reals, except when it is explicitly stated otherwise. But it is clear from the statement of Theorem 1.1 that if A_0 and A_1 happen to be complex Banach spaces, then the value of $\gamma(\vec{A})$ will be the same, independently of whether we consider the underlying scalar field to be $\mathbb{R}$ or $\mathbb{C}$. For a related comment see Remark 2.6.

Our main goal in this paper is to calculate the exact value of, and obtain new estimates for $\gamma(\vec{A})$ for some particular "natural" choices of the couple $\vec{A}$. Some of the auxiliary results which we obtain en route to this goal may perhaps also be useful in the future for other purposes, including the determination of $\gamma(\vec{A})$ for other couples.

Theorem 1.1 is one of the most important and useful results in real interpolation theory, and potentially also has interesting applications beyond that theory, for example in the study of various kinds of moduli of continuity. In its applications so far, the precise value of $\gamma(\vec{A})$ does not seem to play a crucial role. However, as has turned out to be the case with other important theorems in analysis, we believe that searching for optimal constants, and thus optimal proofs, can also enhance our general understanding of this very significant result.

It is known (cf. [11]) that

$$1 \leqslant \gamma(\vec{A}) \leqslant 3 + 2\sqrt{2} \approx 5.8284 \tag{1.3}$$

for every non-zero Banach couple $\vec{A}$.

It has been shown [10] that, in the case where $\vec{A}$ is a non-zero couple of Banach lattices (or complexified Banach lattices) of measurable functions on the same underlying measure space, the estimate (1.3) can be sharpened to

$$1 \leqslant \gamma(\vec{A}) \leqslant 4.$$

A number of couples $\vec{A}$ are known to be *exactly K-divisible*, i.e. to have the property that $\gamma(\vec{A}) = 1$. These include (L^1, L^∞) and the "weighted" L^p couples $(L^1_{w_0}, L^1_{w_1})$ and $(L^\infty_{w_0}, L^\infty_{w_1})$, for all choices of weight functions w_0 and w_1. The proof that $\gamma(\vec{A}) = 1$ for the first of these couples can be found in [16]. It also follows from an obvious generalization of the proof of Lemma 5.2 of [15, p. 44]. The proof for the latter two couples is contained in Proposition 3.2.13 of [7, p. 335]. Let us also mention another collection of trivial examples of exactly K-divisible couples. These are the non-zero couples $\vec{A} = (A_0, A_1)$ for which $A_0 = A_1$ isometrically. (For such a couple, every element $a \in A_0 + A_1$ satisfies $K(t, a; \vec{A}) = \min\{1, t\}\|a\|_{A_0}$. So, if a satisfies (1.1) and we choose $a_n = \frac{\phi_n(1)}{\sum_{m=1}^{\infty} \phi_m(1)} a$ for each $n \in \mathbb{N}$, then it is obvious that we obtain (1.2) with $C_\infty = 1$ when $t = 1$, and consequently also for all $t > 0$.)

On the other hand it is also known that $\gamma(\vec{A}) > 1$ for certain couples $\vec{A}$. The first example to be given of such a couple was $\vec{A} = (C, C^1)$, studied by Krugljak in [18]. Subsequently Podogova [20] showed that this same couple satisfies $\gamma(\vec{A}) \geqslant \frac{3+2\sqrt{2}}{1+2\sqrt{2}} \approx 1.5224$. As announced in [21], Pavel Shvartsman has produced a different and much simpler example of a couple $\vec{S} = (S_0, S_1)$ whose 2-term K-divisibility constant satisfies $\gamma_2(\vec{S}) = \frac{3+2\sqrt{2}}{1+2\sqrt{2}}$. He takes S_0 to be $\mathbb{R}^2$ equipped with the ℓ^∞-norm and S_1 to be a one-dimensional subspace of $\mathbb{R}^2$ whose unit ball is a line segment which makes an angle of $\frac{\pi}{8}$ with one of the coordinate axes. Furthermore, Shvartsman shows that this couple is "extremal" among all couples $\vec{A} = (A_0, A_1)$ satisfying $A_j \subset \mathbb{R}^2$ for $j = 0, 1$, in the sense that all such couples satisfy $\gamma_2(\vec{A}) \leqslant \frac{3+2\sqrt{2}}{1+2\sqrt{2}}$. It will follow from one of our results in this paper that $\gamma(\vec{S}) \leqslant 2\sqrt{2/3} \approx 1.6330$, and thus that the exact value of $\gamma(\vec{S})$ lies somewhere in the interval $(1.52, 1.64)$.

Apparently, neither (C, C^1) nor Shvartsman's finite-dimensional couple can be realized as couples of Banach lattices on a measure space. But it turns out that there also exist couples of lattices whose K-divisibility constant is bigger than 1. The first examples of such couples were found in [12]. They are somewhat "exotic" couples $\vec{A} = (A_0, A_1)$ where the spaces A_0 and A_1 are both contained in $\mathbb{R}^3$. They each satisfy $\gamma(\vec{A}) > 1$ as a consequence of the fact that they do not possess another property, *almost exact monotonicity*, which is defined on p. 30 of [12].

In this paper we deal with what could be considered two of the simplest, "nicest" and most "natural" couples among those which are not already known to be exactly K-divisible, namely a couple $\vec{H} = (H_0, H_1)$ of Hilbert spaces, and the lattice couple (L^2, L^∞). In addition to its other good properties, (H_0, H_1) is known, as shown in [2], to be an exact Calderón couple. (L^2, L^∞) is also a Calderón couple [19] and the optimal decomposition for obtaining its K-functional exactly is quite simple to describe. But it turns out, perhaps rather surprisingly, that neither of these couples are exactly K-divisible in general, and one can even find two-dimensional versions of each of these couples for which exact K-divisibility does not hold.

The paper is organized as follows: In Section 2 we recall some definitions and collect some general preliminary results which will be needed in other sections. In Section 3 we find the exact value of $\gamma(\vec{Y})$ where $\vec{Y}$ is the simplest non-trivial version of a couple of Hilbert spaces. Our result is that $\gamma(\vec{Y}) = 2/\sqrt{3}$. After considering various generalizations of this result, we consider all other couples of (real) Hilbert spaces which are contained in $\mathbb{R}^2$, and we prove a (rather more crude) upper estimate for their K-divisibility constants, namely $\gamma(\vec{G}) < \sqrt{2}$.

Finally, in Section 4 we consider the couple (L^2, L^∞) and, in particular, the case where the underlying measure space consists of two atoms of equal measure, i.e. the two-dimensional couple $\vec{X} = (\ell^2_2, \ell^\infty_2)$. It turns out to be quite easy to show that $\vec{X}$ is an exact Calderón couple and that $\gamma(\vec{X}) > 1$. But the determination of the exact value of $\gamma(\vec{X})$ is a much longer and as yet unfinished story. We obtain some (rather complicated) equations which in principle could be solved to obtain the exact value of $\gamma(\vec{X})$. Numerical experiments suggest that maybe $\gamma(\vec{X})$ is approximately equal to 1.03. The sharpest estimates which we have are $1 < \gamma(\vec{X}) < \frac{4+3\sqrt{2}}{4+2\sqrt{2}} \approx 1.2071$.

These examples demonstrate that there are in general no tight connections between the properties of being an exactly K-divisible couple and of being an exact Calderón couple.

For the reader's convenience, we list and indicate the sizes of some of the numerical constants which appear frequently in the paper. We have that

$$\gamma(\vec{Y}) = 2/\sqrt{3} \approx 1.1547, \qquad \gamma_2(\vec{S}) = \frac{3+2\sqrt{2}}{1+2\sqrt{2}} \approx 1.5224,$$

$$\gamma(\vec{S}) \leqslant 2\sqrt{2/3} \approx 1.6330, \qquad \gamma(\vec{X}) < \frac{4+3\sqrt{2}}{4+2\sqrt{2}} \approx 1.2071.$$

2. Some definitions and general preliminary results

For the basic notions of the real method of interpolation, we refer, e.g. to [4], [5] or [7]. For any given Banach couple $\vec{A} = (A_0, A_1)$, we let $A_j^\sim$ denote the *Gagliardo completion* of A_j, $j = 0, 1$, i.e. the Banach space of elements a of $A_0 + A_1$ which are limits in $A_0 + A_1$ norm of bounded sequences in A_j or, equivalently, for which the norm $\|a\|_{A_j^\sim} = \sup_{t>0} K(t, a; \vec{A})/t^j$ is finite. Obviously $A_0^\sim + A_1^\sim = A_0 + A_1$. We also recall that the couple $\vec{A} = (A_0, A_1)$ and the corresponding couple of its Gagliardo completions $\vec{A}^\sim = (A_0^\sim, A_1^\sim)$ have identical K-functionals, i.e. $K(t, a; \vec{A}) = K(t, a; \vec{A}^\sim)$ for all $a \in A_0 + A_1$ and all $t > 0$. Consequently we also have $\gamma(\vec{A}) = \gamma(\vec{A}^\sim)$.

There is a close connection between K-divisibility and couples of weighted L^1 spaces which we wish to exploit. Our point of departure is the following lemma.

Lemma 2.1. *Let $\vec{A} = (A_0, A_1)$ be an arbitrary Banach couple and let a be an arbitrary element of $A_0 + A_1$. Then there exist a measure space $(\Omega, \mathcal{S}, \mu)$ and measurable functions $w_j : \Omega \to (0, \infty]$ for $j = 0, 1$ and a measurable function $f_a : \Omega \to [0, \infty)$ such that $K(t, a; \vec{A}) = K(t, f_a; \vec{P})$ for all $t > 0$, where $\vec{P}$ is the couple of weighted L^1 spaces $\vec{P} = (L^1_{w_0}(\mu), L^1_{w_1}(\mu))$.*

The straightforward proof of this result, which uses [5, Lemma 5.4.3, p. 117], can be found in [9, pp. 46–47]. It should not be overlooked that the weight functions w_0 and w_1 in Lemma 2.1 have the slightly exotic property that they are permitted to assume the value $+\infty$. Since every function in $L^1_{w_0}(\mu)+L^1_{w_1}(\mu)$ vanishes a.e. on the set where $w_0 = w_1 = \infty$, we always can and will assume that this set is empty. We also mention that the proof in [9] shows that $(\Omega, \mathcal{S}, \mu)$ and w_0 and w_1 can be chosen rather simply and quite explicitly, and we can also, for example, arrange things so that f_a is a constant function.

It turns out that for each $\vec{A}$ and each $a \in A_0 + A_1$ and each corresponding $\vec{P}$ and f_a with the properties just specified, there exists a bounded linear operator $T : \vec{P} \to \vec{A}^{\sim}$ such that $a = Tf_a$. Let $\mathcal{T}_a$ denote the set of all such operators T for some given choice of a and f_a. Then it turns out that

$$\gamma(\vec{A}) = \sup_{a\in A_0+A_1} c_a \quad \text{where } c_a = c_a(\vec{A}) := \inf_{T\in\mathcal{T}_a} \|T\|_{\vec{P}\to\vec{A}^{\sim}}. \tag{2.1}$$

This formula, whose proof will be briefly recalled below, turns out to be particularly suitable for our calculations of K-divisibility constants in this paper.

It is sometimes convenient to re-express (2.1) slightly differently. For $\vec{A}$, a, $\vec{P}$ and f_a as above, let Λ_a be the set of linear operators $T : \vec{P} \to \vec{A}^{\sim}$ with $\|T\|_{\vec{P}\to\vec{A}^{\sim}} \leqslant 1$ such that $Tf_a = \lambda a$ for some positive number $\lambda = \lambda_T$. Then obviously (2.1) is the same as

$$\gamma(\vec{A}) = \sup_{a\in A_0+A_1} \left(\inf_{T\in\Lambda_a} \frac{1}{\lambda_T} \right). \tag{2.2}$$

Remark 2.2. Clearly $\mathcal{T}_{ta} = \mathcal{T}_a$ and so $c_{ta} = c_a$ for all scalars $t \neq 0$. Furthermore, if, as is the case for most couples considered in the paper, A_0 and A_1 are both Banach lattices of measurable functions on the same underlying measure space, then it is easy to see that, in the formula (2.1), the supremum can be replaced by the supremum over all non-negative functions a in $A_0 + A_1$.

Indeed, we have for every $a \in A_0 + A_1$ that $c_a = c_{|a|}$.

At first sight it seems that there could be some ambiguity in (2.1), because the set $\mathcal{T}_a$ depends on our particular choices of the measure space $(\Omega, \mathcal{S}, \mu)$ and the associated functions f_a, w_0 and w_1. The key to showing that in fact there is no such ambiguity is the theorem of Sedaev–Semenov [23] (see [13] for an alternative proof) or, more precisely, the generalization of that theorem [9, Theorem 3, pp. 47–49] to the case of weight functions which are permitted to take the value $+\infty$. It follows immediately from that theorem, that if $(\Xi, \mathcal{Y}, \sigma)$ is a second measure space and v_0 and v_1 are weight functions and g_a is a non-negative measurable function such that $K(t, g_a; L^1_{v_0}(\sigma), L^1_{v_1}(\sigma)) = K(t, f_a; L^1_{w_0}(\mu), L^1_{w_1}(\mu))$ for all $t > 0$ then, for each $\epsilon > 0$, there exist two linear operators $U : (L^1_{v_0}(\sigma), L^1_{v_1}(\sigma)) \to (L^1_{w_0}(\mu), L^1_{w_1}(\mu))$ and $V : (L^1_{w_0}(\mu), L^1_{w_1}(\mu)) \to (L^1_{v_0}(\sigma), L^1_{v_1}(\sigma))$ which satisfy $Ug_a = f_a$, $Vf_a = g_a$, $\|U\|_{(L^1_{v_0}(\sigma),L^1_{v_1}(\sigma))\to(L^1_{w_0}(\mu),L^1_{w_1}(\mu))} \leqslant 1+\epsilon$ and $\|V\|_{(L^1_{w_0}(\mu),L^1_{w_1}(\mu))\to(L^1_{v_0}(\sigma),L^1_{v_1}(\sigma))} \leqslant 1+\epsilon$.

By composing the operators U and V with other suitable operators, we readily see that the quantity $\inf_{T\in\mathcal{T}_a} \|T\|_{\vec{P}\to\vec{A}^{\sim}}$ is independent of the choices of the measure space, weight functions and the function f_a.

For the convenience of the reader who may not be familiar with these details, we mention that the fact that $\mathcal{T}_a$ is non-empty and the formula (2.1) are both obtained by considering the following theorem which, as we shall explain, is intimately related, in fact equivalent, to Theorem 1.1 (cf. [14, Proposition 1.40]).

Theorem 2.3. *Let $\vec{A} = (A_0, A_1)$ be an arbitrary Banach couple. Then there exist constants M_1, M_2 and M_3, depending only on $\vec{A}$, with, respectively, the following properties*:

(i) *For each $a \in A_0 + A_1$, there exists a sequence $\{a_\nu\}_{\nu\in\mathbb{Z}}$ of elements in $A_0^{\sim} \cup A_1^{\sim}$ which satisfies $a = \sum_{\nu\in\mathbb{Z}} a_\nu$ (convergence in $A_0 + A_1$ norm) and also*

$$\sum_{\nu\in\mathbb{Z}} \min\{\|a_\nu\|_{A_0^{\sim}}, t\|a_\nu\|_{A_1^{\sim}}\} \leqslant M_1 K(t, a; \vec{A}) \quad \textit{for all } t > 0. \tag{2.3}$$

(ii) *Let w_0 and w_1 be arbitrary weight functions on an arbitrary measure space $(\Omega, \mathcal{S}, \mu)$. Let $\vec{P}$ be the couple of weighted L^1 spaces $\vec{P} = (L^1_{w_0}(\mu), L^1_{w_1}(\mu))$. Suppose that the elements $a \in A_0 + A_1$ and $f \in L^1_{w_0} + L^1_{w_1}$ satisfy*

$$K(t, a; \vec{A}) \leqslant K(t, f; \vec{P}) \quad \textit{for all } t > 0. \tag{2.4}$$

Then there exists a bounded linear operator $T : \vec{P} \to \vec{A}^{\sim}$ such that $\|T\|_{\vec{P}\to\vec{A}^{\sim}} \leqslant M_2$ and $Tf = a$.

(iii) *Suppose that $(\Omega, \mathcal{S}, \mu)$, w_0, w_1, f and a are exactly as in part* (ii), *except that instead of* (2.4) *they satisfy*

$$K(t, a; \vec{A}) = K(t, f; \vec{P}) \quad \textit{for all } t > 0.$$

Then there exists a bounded linear operator $T : \vec{P} \to \vec{A}^{\sim}$ such that $\|T\|_{\vec{P}\to\vec{A}^{\sim}} \leqslant M_3$ and $Tf = a$.

In fact the infima of all constants M_1, M_2 and M_3 satisfying (i), (ii) *and* (iii) *respectively, coincide, and they all equal $\gamma(\vec{A})$, the infimum of the constants C_∞ for which Theorem* 1.1 *holds.*

For a proof of part (ii) of this theorem, which uses Theorem 1.1 and gives the value $M_2 = C_\infty + \epsilon$ for any choice of $\epsilon > 0$, see [7, Theorem 4.4.12, pp. 586–588]. We mention in passing that part (ii) has an important and immediate consequence. It provides a simple description of all relative interpolation spaces for operators mapping from any weighted L^1 couple into any Banach couple $\vec{A}$ which satisfies $A_j^\sim = A_j$ for $j = 0, 1$.

Part (i), also known as the "strong fundamental lemma", is proved in [9, Theorem 4, pp. 54–59] for $M_1 \approx 8$ and, with a better constant $M_1 \approx 3 + 2\sqrt{2}$, in [11, pp. 73–77]. Cf. also [10] for more explicit versions of some of the steps of the proof in [11]. (Note that in (2.3) we adopt the conventions that $\|a_\nu\|_{A_j^\sim} = \infty$ if $a \notin A_j^\sim$ and that $\min\{\alpha, \infty\} = \min\{\infty, \alpha\} = \alpha$ for every $\alpha \in \mathbb{R}$.)

Part (ii) can be deduced from part (i), and with $M_2 = M_1 + \epsilon$ for any choice of $\epsilon > 0$. This can be done, using (an obvious modification of) an argument which appears in [9, pp. 54–55] cf. also [15, Theorem 4.8, p. 38]. Moreover, this result, and also part (iii), are also both valid in the case where either or both of the weight functions w_0 and w_1 are permitted to take the value $+\infty$ on some subsets of Ω. The proof in [9] makes use of the generalized version [9, Theorem 3, p. 47] of the Sedaev–Semenov theorem already mentioned above. (The Sedaev–Semenov theorem is also the main, perhaps only, ingredient of the "obvious modification" mentioned above.)

The connection between parts (ii) and (iii) is a simple matter. Obviously (ii) implies (iii) with $M_3 = M_2$. On the other hand we can also easily obtain that (iii) implies (ii) with $M_2 = M_3 + \epsilon$ for any choice of $\epsilon > 0$. This is done by first using Lemma 2.1 to obtain f_a and then using the generalized version of the Sedaev–Semenov theorem to find a linear map U between appropriate couples of weighted L^1 spaces, which satisfies $Uf = f_a$ and has norm arbitrarily close to 1.

Theorem 1.1, with $C_\infty = M_2$ can be deduced from part (ii) of Theorem 2.3, again using arguments from [9, pp. 54–55] and using the more general version where the weight functions are permitted to take infinite values.

Conversely, as mentioned in [11, p. 71] and shown more explicitly in [14, Proposition 1.40], it is also possible to deduce part (i) (and consequently also part (ii)) of Theorem 2.3 from Theorem 1.1, with $M_1 = C_\infty + \epsilon$ for any choice of $\epsilon > 0$.

It should be noted that part (iii) of the above theorem, together with the connections described above between the constants M_1, M_2 and M_3 for which parts (i)–(iii) of the theorem hold, give us the formula (2.1).

Remark 2.4. Let $a \in A_0 + A_1$ be a fixed element. By an adaption of the proof of Theorem 2.3, one can show that the infima of the constants M_1, M_2 and M_3 fulfilling the conditions (i)–(iii) for this particular choice of a, coincide, and their common value is $c_a(\vec{A})$.

For most couples $\vec{A} = (A_0, A_1)$ which we study in this paper, A_0 and A_1 are both finite-dimensional. For such couples it is clear that $A_j^\sim = A_j$ isometrically for $j = 0, 1$. It is also helpful to know, as the following lemma shows, that, for such couples, the infimum $\inf_{T \in \mathcal{T}_a} \|T\|_{\vec{P} \to \vec{A}}$ appearing in (2.1) is actually attained for each fixed element a. This of course implies that the infimum $\inf_{T \in \Lambda_a} 1/\lambda_T$ in (2.2) is also attained for each a. We will refer to any operator T for which this latter infimum is attained as an ***optimal element of*** Λ_a. Obviously such an operator satisfies $\|T\|_{\vec{P} \to \vec{A}} = 1$.

Lemma 2.5. *Let $\vec{F} = (F_0, F_1)$ and $\vec{A} = (A_0, A_1)$ be Banach couples and suppose that $A_0 + A_1$ is a finite-dimensional space. Let a and f be arbitrary fixed elements of $A_0 + A_1$ and $F_0 + F_1$ respectively. Suppose that the class $\mathcal{T}_a$ of all bounded linear operators $T : \vec{F} \to \vec{A}$ which satisfy $Tf = a$ is non-empty. Then there exists an operator $S \in \mathcal{T}_a$ such that $\|S\|_{\vec{F} \to \vec{A}} = \inf_{T \in \mathcal{T}_a} \|T\|_{\vec{F} \to \vec{A}}$.*

Proof. Let N be the dimension of $A_0 + A_1$ and let $\{e_k\}_{k=1}^N$ be a basis of $A_0 + A_1$. Then every bounded operator $T : F_0 + F_1 \to A_0 + A_1$ defines and can be defined by a collection $\lambda_1, \lambda_2, \ldots, \lambda_N$ of N linear bounded linear functionals on $F_0 + F_1$, via the formula $Tg = \sum_{k=1}^N \lambda_k(g) e_k$ for each $g \in F_0 + F_1$. Consider a sequence of elements $\{T_n\}_{n \in \mathbb{N}}$ in $\mathcal{T}_a$ such that $\|T_n\|_{\vec{F} \to \vec{X}} \leqslant c_a + 1/n$, where $c_a = \inf_{T \in \mathcal{T}_a} \|T\|_{\vec{F} \to \vec{A}}$. Let $\lambda_{n,k}$ denote the bounded linear functional on $F_0 + F_1$ defined for each $n \in \mathbb{N}$ and each $k \in \{1, 2, \ldots, N\}$, such that $T_n g = \sum_{k=1}^N \lambda_{n,k}(g) e_k$ for each $g \in F_0 + F_1$. Now let us define the operator S by

$$Sg = \sum_{k=1}^N \lambda_{*,k}(g) e_k \quad \text{for each } g \in F_0 + F_1,$$

where the N linear functionals $\lambda_{*,1}, \lambda_{*,2}, \ldots, \lambda_{*,N}$ are given by

$$\lambda_{*,k}(g) = B\left(\left\{\lambda_{n,k}(g)\right\}_{n \in \mathbb{N}}\right)$$

for each $g \in F_0 + F_1$, where $B \in (\ell^\infty)^*$ is a Banach limit (i.e. an element of $(\ell^\infty)^*$ which satisfies $|B(\{u_n\}_{n \in \mathbb{N}})| \leqslant \limsup_{n \to \infty} |u_n|$ for all $\{u_n\}_{n \in \mathbb{N}} \in \ell^\infty$ and also $B(\{u_n\}_{n \in \mathbb{N}}) = \lim_{n \to \infty} u_n$ for every convergent sequence $\{u_n\}_{n \in \mathbb{N}}$). It easy to see that each sequence $\{\lambda_{n,k}(g)\}_{n \in \mathbb{N}}$ is indeed in ℓ^∞ and it is straightforward, if a little tedious, to verify that the operator S has all the required properties. We leave these matters to the reader. □

Remark 2.6. For all the couples $\vec{A}=(A_0,A_1)$ considered in this paper, A_0 and A_1 are both Banach lattices of real valued measurable functions with the same underlying measure space. As usual, we can define the complexification of such a lattice A_j to be the space, which we may denote by $A_j^{\mathbb{C}}$, consisting of all complex valued measurable functions g such that $|g|\in A_j$, with the obvious norm. It is easy to see that the complexified lattice couple $\vec{A}^{\mathbb{C}}=(A_0^{\mathbb{C}},A_1^{\mathbb{C}})$ satisfies $\gamma(\vec{A}^{\mathbb{C}})=\gamma(\vec{A})$. (Use the fact that for any function $a\in A_0^{\mathbb{C}}+A_1^{\mathbb{C}}$ we have $K(t,a;\vec{A}^{\mathbb{C}})=K(t,|a|;\vec{A})$.)

3. On the *K*-divisibility constant for Hilbert couples

3.1. The *K*-divisibility constant for the couple $\vec{Y}=(Y_0,Y_1)=(\ell_2^2,\ell_1^2)$

The purpose of this subsection is to prove the following theorem.

Theorem 3.1. *Let $\vec{Y}=(Y_0,Y_1)$ be the Banach couple of subspaces of $\mathbb{R}^2$ obtained by taking the unit ball of Y_0 to be the disk $\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\leqslant 1\}$ and the unit ball of Y_1 to be the line segment $\{(x,0)\in\mathbb{R}^2:\ -1\leqslant x\leqslant 1\}$. Then the K-divisibility constant of the couple $\vec{Y}$ is given by*

$$\gamma(\vec{Y})=\frac{2}{\sqrt{3}}. \tag{3.1}$$

Proof. Consider the point $\alpha=(\cos a,\sin a)\in Y_0+Y_1$ where $a\in[0,2\pi)$. Let E_a be the set consisting of every number which is the norm $\|T\|_{\vec{P}\to\vec{Y}}$ of some bounded linear operator T from some couple $\vec{P}$ of weighted L^1 spaces into $\vec{Y}$, which satisfies $Tf=\alpha$ for some element $f\in P_0+P_1$ for which

$$K(t,f;\vec{P})=K(t,\alpha;\vec{Y})\quad\text{for all } t>0. \tag{3.2}$$

Note that the weight functions w_0 and w_1 used in the definition of P_0 and P_1 are permitted to assume the value $+\infty$ on some sets of positive measure. We shall explicitly need this option here.

Let $c_a=\inf E_a$. It follows from Remark 2.2 that $\gamma(\vec{Y})=\sup_{a\in[0,\pi/2]}c_a$. We claim that in fact

$$\gamma(\vec{Y})=\sup_{a\in(0,\pi/2)}c_a. \tag{3.3}$$

To show (3.3) we first observe that, since $K(t,Tf;\vec{Y})\leqslant\|T\|_{\vec{P}\to\vec{Y}}K(t,f;\vec{P})$ for all $t>0$ and for every bounded operator $T:\vec{P}\to\vec{Y}$, we must have $c_a\geqslant 1$ for every $a\in[0,\pi/2]$. It turns out to be rather easy to show that $c_a\leqslant 1$ in the two special cases, $a=0$ and $a=\pi/2$, and this will of course imply (3.3).

In the case where $a=0$, i.e., $\alpha=(1,0)$, we use a very simple couple $\vec{P}$ where the underlying measure space consists of a single point b which has measure 1 and $\|h\|_{P_0}=\|h\|_{P_1}=|h(b)|$ for every $h\in P_0+P_1$. (I.e., $w_0(b)=w_1(b)=1$.) We also use the "function" $f\in P_0+P_1$ defined by $f(b)=1$ which clearly satisfies

$$K(t,f;\vec{P})=\min\{1,t\}=K\big(t,(1,0);\vec{Y}\big)\quad\text{for all } t>0.$$

Then we use the operator T defined by $T(h)=(h(b),0)$ for all $h\in P_0+P_1$ to show that $c_0\leqslant 1$.

In the case where $a=\pi/2$, i.e., $\alpha=(0,1)$, it is convenient, once again, to use an underlying measure (Ω,Σ,μ) space containing (at least) one point b which is an atom of measure 1. But this time the weight functions w_0 and w_1 for which $P_j=L^1_{w_j}(\mu)$ should be chosen to satisfy $w_0(b)=1$ and $w_1(b)=+\infty$. This means that every function h in P_1 satisfies $h(b)=0$ and so the linear map T defined by $Th=(0,h(b))$ maps P_1 into Y_1 with norm 0 and P_0 into Y_0 with norm 1. Furthermore the function $f=\chi_{\{b\}}$ satisfies $Tf=(0,1)$ and $K(t,f;\vec{P})=1=K(t,(0,1);\vec{Y})$ for all $t>0$. This shows that $c_{\pi/2}\leqslant 1$ and so completes the proof of (3.3).

In the light of the preceding calculations it remains to calculate or estimate c_a for values of $a\in(0,\pi/2)$. So let us indeed fix $a\in(0,\pi/2)$ and set $\alpha=(\cos a,\sin a)=(\alpha_1,\alpha_2)$. It is easy to see that the error functional $E(t,\alpha;\vec{Y})=\inf\{\|\alpha-\beta\|_{Y_0}:\ \beta\in Y_1,\ \|\beta\|_{Y_1}\leqslant t\}$ is given by the formula

$$E(t,\alpha;\vec{Y})=\begin{cases}\sqrt{(t-\alpha_1)^2+\alpha_2^2}, & t\in[0,\alpha_1],\\ \alpha_2, & t>\alpha_1.\end{cases}$$

Now we will describe a particular couple of weighted L^1 spaces $\vec{P}=(P_0,P_1)$ on the (non-empty) interval $[0,\alpha_1]$, for which the function $f=\chi_{[0,\alpha_1]}$ satisfies (3.2). Once again we use the fact that (3.2) is equivalent to

$$E(t,f;\vec{P})=E(t,\alpha,\vec{Y})\quad\text{for all } t>0. \tag{3.4}$$

To make (3.4) hold, we choose a measure μ on $[0,\alpha_1]$ which coincides with Lebesgue measure on $[0,\alpha_1)$ and such that the singleton set $\{\alpha_1\}$ has measure $\mu(\{\alpha_1\})=1$. Then we take $P_j = L_{w_j}([0,\alpha_1],\mu)$ for $j=0,1$, where the weight functions w_0 and w_1 are defined by

$$w_0(t)=\begin{cases}-\frac{d}{dt}E(t,\alpha;\vec{Y})=\frac{\alpha_1-t}{\sqrt{(\alpha_1-t)^2+\alpha_2^2}}, & t\in[0,\alpha_1),\\ \alpha_2, & t=\alpha_1\end{cases}$$

and

$$w_1(t)=\begin{cases}1, & t\in[0,\alpha_1),\\ +\infty, & t=\alpha_1.\end{cases}$$

Since w_0 is decreasing on $[0,\alpha_1)$ it is easy to obtain that

$$E(t,f,\vec{P})=\int\limits_{[\min(t,\alpha_1),\alpha_1]} w_0\,d\mu=\int\limits_{[t,\infty)\cap[0,\alpha_1)} w_0\,d\mu+\alpha_2$$

for each $t>0$ which immediately also gives us (3.4) and (3.2).

Let us now define E_a^* to be the subset of E_a consisting of the numbers $\|T\|_{\vec{P}\to\vec{Y}}$ obtained in the special case where $\vec{P}$ is the particular couple

$$\vec{P}=\big(L^1_{w_0}\big([0,\alpha_1],\mu\big),L^1_{w_1}\big([0,\alpha_1],\mu\big)\big)$$

which we have just defined, and the function f for which $Tf=\alpha$ is given by $f=\chi_{[0,\alpha_1]}$. In view of the generalized version of the Sedaev–Semenov theorem in [9], it is clear that c_a is also the infimum of the set E_a^*.

Any bounded linear operator $T:\vec{P}\to\vec{Y}$ for this particular choice of $\vec{P}$ must be given by the formula

$$Th=\bigg(\int\limits_{[0,\alpha_1)} g_1(\xi)h(\xi)\,d\xi+\beta_1 h(\alpha_1),\int\limits_{[0,\alpha_1)} g_2(\xi)h(\xi)\,d\xi+\beta_2 h(\alpha_1)\bigg) \tag{3.5}$$

for all $h\in P_0+P_1$. Here g_1 and g_2 are suitable bounded measurable functions on $[0,\alpha_1)$ and β_1 and β_2 are real numbers. For all $h\in P_1$ we have $h(\alpha_1)=0$. But all such functions h must also satisfy $\int_{[0,\alpha_1)} g_2(\xi)h(\xi)\,d\xi+\beta_2 h(\alpha_1)=0$. Consequently $g_2=0$ a.e. on $[0,\alpha_1)$. Thus the norm $\|T\|_{P_1\to Y_1}$ equals $\|g_1\|_{L^\infty[0,\alpha_1)}$. The norm $\|T\|_{P_0\to Y_0}$ is the supremum of

$$\theta_1\int\limits_{[0,\alpha_1]}(g_1\chi_{[0,\alpha_1)}+\beta_1\chi_{\{\alpha_1\}})h\,d\mu+\theta_2\beta_2 h(\alpha_1)=\int\limits_{[0,\alpha_1]}\big(\theta_1 g_1\chi_{[0,\alpha_1)}+(\theta_1\beta_1+\theta_2\beta_2)\chi_{\{\alpha_1\}}\big)h\,d\mu$$

as h ranges over the unit ball of P_0 and (θ_1,θ_2) ranges over the unit circle. Let us first calculate the supremum, for a fixed choice of (θ_1,θ_2), as h ranges over the unit ball of P_0. The standard duality between L^1 and L^∞ gives us that this supremum equals

$$\bigg\|\frac{\theta_1 g_1\chi_{[0,\alpha_1)}+(\theta_1\beta_1+\theta_2\beta_2)\chi_{\{\alpha_1\}}}{w_0}\bigg\|_{L^\infty([0,\alpha_1],\mu)}=\max\bigg\{\theta_1\operatorname*{ess\,sup}_{\xi\in[0,\alpha_1)}\bigg|\frac{g_1(\xi)}{w_0(\xi)}\bigg|,\frac{|\theta_1\beta_1+\theta_2\beta_2|}{\alpha_2}\bigg\}. \tag{3.6}$$

We now claim that

$$\|T\|_{P_0\to Y_0}=\max\bigg\{\operatorname*{ess\,sup}_{\xi\in[0,\alpha_1)}\bigg|\frac{g_1(\xi)}{w_0(\xi)}\bigg|,\frac{\sqrt{\beta_1^2+\beta_2^2}}{\alpha_2}\bigg\}. \tag{3.7}$$

This is because the expression in (3.6) equals the expression on the right side of (3.7) for a suitable choice of (θ_1,θ_2) on the unit circle (either $(1,0)$ or $(\frac{\beta_1}{\sqrt{\beta_1^2+\beta_2^2}},\frac{\beta_2}{\sqrt{\beta_1^2+\beta_2^2}})$). Furthermore it is dominated by the expression on the right side of (3.7) for all other points (θ_1,θ_2) on the unit circle.

Since $w_0(\xi)<1$ for all $\xi\in[0,\alpha_1)$, we have that

$$\begin{aligned}\|T\|_{P_1\to Y_1}&=\|g_1\|_{L^\infty[0,\alpha_1)}\\&=\operatorname*{ess\,sup}_{[0,\alpha_1)}\big|g_1(\xi)\big|\leqslant\operatorname*{ess\,sup}_{\xi\in[0,\alpha_1)}\bigg|\frac{g_1(\xi)}{w_0(\xi)}\bigg|.\end{aligned}$$

This means that the norm $\|T\|_{\vec{P}\to\vec{Y}}$ is also given by the expression on the right side of (3.7).

Of course here we are only concerned with those operators T for which $T\chi_{[0,\alpha_1]}=(\alpha_1,\alpha_2)$, i.e.

$$\int\limits_{[0,\alpha_1)} g_1(\xi)\,d\xi+\beta_1=\alpha_1\quad\text{and}\quad\beta_2=\alpha_2. \tag{3.8}$$

By Lemma 2.5 there exists such an operator T which satisfies $\|T\|_{\vec{P}\to\vec{Y}} = c_a$.

Evidently the functions g_1 and numbers β_1 and β_2 which are used in the formula defining T must satisfy $|g_1(\xi)| \leqslant c_a w_0(\xi)$ for a.e. $\xi \in [0,\alpha_1)$ and $\sqrt{\beta_1^2+\beta_2^2} \leqslant c_a\alpha_2$. Consequently, substituting from (3.8), we have

$$\alpha_1 = \int_{[0,\alpha_1)} g_1(\xi)\,d\xi + \beta_1 \leqslant \int_{[0,\alpha_1)} c_a w_0(\xi)\,d\xi + \sqrt{c_a^2\alpha_2^2 - \beta_2^2}$$
$$= c_a\left(\sqrt{\alpha_1^2+\alpha_2^2} - \alpha_2\right) + \alpha_2\sqrt{c_a^2-1} = c_a(1-\alpha_2) + \alpha_2\sqrt{c_a^2-1}.$$

In the special case where $a = \pi/6$, i.e. when $\alpha_1 = \sqrt{3}/2$ and $\alpha_2 = 1/2$, the previous inequalities immediately imply that

$$\sqrt{3} \leqslant c_{\pi/6} + \sqrt{c_{\pi/6}^2 - 1}.$$

This is false if $c_{\pi/6} < 2/\sqrt{3}$. I.e., we have shown that

$$c_{\pi/6} \geqslant 2/\sqrt{3}. \tag{3.9}$$

We shall now prove that $c_a \leqslant 2/\sqrt{3}$ for all $a \in (0,\pi/2)$. Having chosen such a value of a, we set $\alpha_1 = \cos a$ and $\alpha_2 = \sin a$. Since

$$\alpha_1 + \alpha_2 = \sqrt{\alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2} > \sqrt{\alpha_1^2+\alpha_2^2} = 1,$$

we have that $\frac{\alpha_1}{1-\alpha_2} > 1$. It is clear that the function $\phi(x) := \frac{\alpha_1 - x}{1-\alpha_2}$ decreases from $\frac{\alpha_1}{1-\alpha_2}$ to 1 on the interval $I = [0, \alpha_1 + \alpha_2 - 1]$. This in turn means that the continuous function $\psi(x) := \alpha_2\sqrt{\phi^2(x)-1} - x$ is also decreasing on the same interval. Since $\psi(0) > 0$ and $\psi(\alpha_1+\alpha_2-1) = 1-\alpha_1-\alpha_2 < 0$, there exists a number β_1 in the interior of I such that $\psi(\beta_1) = 0$, i.e.

$$\beta_1 = \alpha_2\sqrt{\left(\frac{\alpha_1-\beta_1}{1-\alpha_2}\right)^2 - 1}. \tag{3.10}$$

We shall use this number in the formula (3.5) to define an operator $T : \vec{P} \to \vec{Y}$ where we choose the other numbers and functions in the formula by setting $g_2(\xi) = 0$ (as we are obliged to do) and also

$$g_1(\xi) = \frac{\alpha_1-\beta_1}{1-\alpha_2} w_0(\xi) \quad \text{for all } \xi \in [0,\alpha_1) \text{ and } \beta_2 = \alpha_2. \tag{3.11}$$

Observe that, with these definitions,

$$T\chi_{[0,\alpha_1]} = \left(\frac{\alpha_1-\beta_1}{1-\alpha_2}(1-\alpha_2) + \beta_1, \alpha_2\right) = (\alpha_1, \alpha_2),$$

i.e. the quantity $C_a := \|T\|_{\vec{P}\to\vec{Y}}$ belongs to E_a^*. In particular, $c_a \leqslant C_a$. But, in view of (3.7) and (3.10), we have

$$C_a = \frac{\alpha_1-\beta_1}{1-\alpha_2}.$$

This in turn can be substituted in (3.10) to give

$$\beta_1 = \alpha_2\sqrt{C_a^2-1}$$

and so

$$C_a = \frac{\alpha_1-\beta_1}{1-\alpha_2} = \frac{\alpha_1 - \alpha_2\sqrt{C_a^2-1}}{1-\alpha_2}.$$

We deduce that

$$C_a + \frac{\alpha_2}{1-\alpha_2}\sqrt{C_a^2-1} = \frac{\alpha_1}{1-\alpha_2}. \tag{3.12}$$

We claim that (3.12) implies that

$$C_a \leqslant 2/\sqrt{3}. \tag{3.13}$$

If this is false, then

$$\frac{\alpha_1}{1-\alpha_2} > \frac{2}{\sqrt{3}} + \frac{\alpha_2}{1-\alpha_2}\sqrt{\frac{4}{3}-1} = \frac{1}{\sqrt{3}}\left(2 + \frac{\alpha_2}{1-\alpha_2}\right)$$

and so $\sqrt{3}\alpha_1 > 2(1-\alpha_2)+\alpha_2 = 2-\alpha_2$. Consequently, $3\alpha_1^2 > 4-4\alpha_2+\alpha_2^2$. Since $\alpha_1^2+\alpha_2^2=1$ it follows that $3-3\alpha_2^2 > 4-4\alpha_2+\alpha_2^2$, i.e. that $4\alpha_2^2-4\alpha_2+1<0$. But this cannot hold for any real number α_2. This contradiction establishes (3.13).

We immediately deduce that $c_a \leqslant 2/\sqrt{3}$ for all $a \in (0,\pi/2)$. Combining this with (3.9) and (3.3) gives (3.1) and completes the proof of the theorem. □

3.2. Generalizations and further remarks

We have the following generalization of Theorem 3.1.

Theorem 3.2. *Let U and V be non-trivial Hilbert spaces and consider the couple $\vec{W}=(U\oplus V, U)$. Then $\gamma(\vec{W})=2/\sqrt{3}$.*

The proof is very similar to the case of $\vec{Y}$. We sketch the changes necessary to make the proof work in the general case.

Each norm one element of $U\oplus V$ can be written in the form $\alpha_1 u+\alpha_2 v$ where $u\in U$ and $v\in V$ are unit vectors and the numbers α_1 and α_2 satisfy $\alpha_1\geqslant 0$, $\alpha_2\geqslant 0$ and $\alpha_1^2+\alpha_2^2=1$. It is easy to see that

$$K(t,\alpha_1 u+\alpha_2 v;\vec{W}) = K(t,\alpha;\vec{Y}) = K(t,f;\vec{P}),\quad t>0,$$

with $f=\chi_{[0,\alpha_1]}$ and $\vec{P}=(L^1_{w_0},L^1_{w_1})$ defined as before. For $w\in W_0+W_1$ let $c_w=c_w(\vec{W})$ be the quantity defined by (2.1). It follows from Remark 2.2 that

$$\gamma(\vec{W})=\sup\{c_{\alpha_1 u+\alpha_2 v}\},$$

the supremum being taken over all points (α_1,α_2) of the unit circle such that $\alpha_1,\alpha_2\geqslant 0$ and all unit vectors $u\in U$, $v\in V$.

For fixed u, v and $\alpha=(\alpha_1,\alpha_2)$ as above we now choose $T:\vec{P}\to\vec{W}$ as

$$Th=\left(\int_{[0,\alpha_1)} g_1(\xi)h(\xi)\,d\xi+\beta_1 h(\alpha_1)\right)u+\alpha_2 h(\alpha_1)v \tag{3.14}$$

where the functions g_1 and the number β_1 are defined by (3.10) and (3.11). Clearly, $Tf=\alpha_1 u+\alpha_2 v$. Moreover, as in the case for $\vec{Y}$, one verifies that this operator T satisfies

$$\|T\|_{\vec{P}\to\vec{W}}=\max\left\{\operatorname*{ess\,sup}_{\xi\in[0,\alpha_1)}\left|\frac{g_1(\xi)}{w_0(\xi)}\right|,\ \frac{\sqrt{\beta_1^2+\alpha_2^2}}{\alpha_2}\right\}.$$

By the reasoning at the end of the proof of Theorem 3.1, we now obtain that $c_{\alpha_1 u+\alpha_2 v}\leqslant\|T\|_{\vec{P}\to\vec{W}}\leqslant 2/\sqrt{3}$, proving that $\gamma(\vec{W})\leqslant 2/\sqrt{3}$.

In order to prove the reverse inequality, we observe that an arbitrary operator $S:\vec{P}\to\vec{W}$ such that $Sf=\alpha_1 u+\alpha_2 v$ can be represented in the form

$$Sh=\left(\int_{[0,\alpha_1)} G_1(\xi)h(\xi)\,d\xi+B_1 h(\alpha_1)\right)\oplus\big(\alpha_2 h(\alpha_1)v\big)$$

where $h\in P_0+P_1$ and $G_1\in L^\infty([0,\alpha_1),U)$ and $B_1\in U$. Putting $g_1(\xi)=(G_1(\xi),u)$ and $\beta_1=(B_1,u)$, we obtain a corresponding operator T of the form (3.14) also satisfying $Tf=\alpha_1 u+\alpha_2 v$ and such that $\|T\|_{\vec{P}\to\vec{W}}\leqslant\|S\|_{\vec{P}\to\vec{W}}$. Now, in the case when $\alpha_1=\sqrt{3}/2$ and $\alpha_2=1/2$, the estimate $\|T\|_{\vec{P}\to\vec{W}}\geqslant 2/\sqrt{3}$ follows exactly as in the case for $\vec{Y}$.

It seems plausible that couples of the above form are extremal amongst all Hilbert couples in the sense that their K-divisibility constant is maximal. Thus we have the following open question.

Question 1. Does $\gamma(\vec{H})\leqslant 2/\sqrt{3}$ hold for every Hilbert couple $\vec{H}$?

For a comment related to this question, see Remark 3.8 below.

We now turn to another consequence of Theorem 3.1. We have the following result:

Theorem 3.3. *Let $\vec{X}=(X_0,X_1)$ be a Banach couple such that X_0 is two-dimensional and X_1 is one-dimensional and $X_1\subset X_0$. Then $\gamma(\vec{X})\leqslant 2\sqrt{2/3}$.*

Note that Shvartsman's couple $\vec{S}$ [21], where S_0 is $\mathbb{R}^2$ equipped with the ℓ_∞-norm and S_1 the one-dimensional subspace of $\mathbb{R}^2$ whose unit ball makes an angle of $\pi/8$ with the positive x-axis, is of the form occurring in Theorem 3.3 and satisfies $\gamma(\vec{S})\geqslant\frac{3+2\sqrt{2}}{1+2\sqrt{2}}$. (At this point it may be helpful to glance back at the table giving the sizes of numerical constants, which appears at the end of Section 1.)

We shall see that Theorem 3.3 can be deduced using the relation (2.1), i.e., $\gamma(\vec{X}) = \sup_{a\in X_0+X_1} c_a(\vec{X})$. To this end, we need a way to keep track of how the numbers c_a change under suitable maps. We will first prove that every so-called "rigid map" leaves c_a unchanged.

Definition 3.4. Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be two Banach couples. A linear operator $T : A_0 + A_1 \to B_0 + B_1$ which is one-to-one, and, for $j = 0$ and $j = 1$, maps A_j onto B_j and satisfies $\|Ta\|_{B_j} = c_j\|a\|_{A_j}$ for all $a \in A_j$ and some positive constant c_j, is called a ***rigid map*** of $\vec{A}$ onto $\vec{B}$. If such a map exists, then we say that $\vec{B}$ is a ***rigid image*** of $\vec{A}$. (This is of course the same as saying that $\vec{A}$ is a rigid image of $\vec{B}$.)

A classical and much used example of two couples which are rigid images of each other, goes back to the paper [25] of Stein and Weiss, where it was pointed out that, in the terminology of Definition 3.4, any couple of weighted L^p spaces $\vec{B} = (L^{p_0}_{w_0}(\Omega,\Sigma,\mu), L^{p_1}_{w_1}(\Omega,\Sigma,\mu))$ on some measure space (Ω,Σ,μ) where $1 \leqslant p_0 < p_1 \leqslant \infty$, is a rigid image of an unweighted couple $\vec{A} = (L^{p_0}(\Omega,\Sigma,\nu), L^{p_1}(\Omega,\Sigma,\nu))$ for some other measure ν on the same measure space.

Fact 3.5. *If $\vec{B}$ is a rigid image of $\vec{A}$ then $\gamma(\vec{B}) = \gamma(\vec{A})$. Furthermore we have that $c_a(\vec{A}) = c_{Ta}(\vec{B})$ for all $a \in A_0 + A_1$, where T is a rigid map of $\vec{A}$ onto $\vec{B}$.*

In order to prove Fact 3.5, we first note that standard arguments show immediately that $K(t, Ta; \vec{B}) = c_0 K(\frac{c_1 t}{c_0}, a; \vec{A})$ for all $t > 0$ and all $a \in A_0 + A_1$.

Put $b = Ta$ and suppose that $K(t, b; \vec{B}) \leqslant \sum_{n=1}^{\infty} \psi_n(t)$ for all $t > 0$, where the functions $\psi_n : (0,\infty) \to (0,\infty)$ are all concave and $\sum_{n=1}^{\infty} \psi_n(1) < \infty$. Then $c_0 K(\frac{c_1 t}{c_0}, a; \vec{A}) \leqslant \sum_{n=1}^{\infty} \psi_n(t)$. Since $\phi_n(t) := c_0^{-1}\psi_n(\frac{c_0 t}{c_1})$ is concave for each n and $\sum_{n=1}^{\infty} \phi_n(1) < \infty$ it follows from Remark 2.4 that, for each $\epsilon > 0$, there exists a sequence of elements $\{a_n\}_{n\in\mathbb{N}}$ in $A_0 + A_1$ such that $a = \sum_{n=1}^{\infty} a_n$ with convergence in $A_0 + A_1$ norm and $K(t, a_n; \vec{A}) \leqslant (c_a(\vec{A}) + \epsilon)\phi_n(t)$ for all $t > 0$ and all $n \in \mathbb{N}$. If we set $b_n = Ta_n$ for each n then it is clear that $b = \sum_{n=1}^{\infty} b_n$ with convergence in $B_0 + B_1$ norm and

$$K(t, b_n; \vec{A}) = c_0 K\left(\frac{c_1 t}{c_0}, a_n; \vec{A}\right) \leqslant \big(c_a(\vec{A}) + \epsilon\big) c_0 \phi_n\left(\frac{c_1 t}{c_0}\right) = \big(c_a(\vec{A}) + \epsilon\big)\psi_n(t)$$

for all $t > 0$ and all $n \in \mathbb{N}$. This shows that $c_b(\vec{B}) \leqslant c_a(\vec{A}) + \epsilon$ for each positive ϵ. It follows that $c_b(\vec{B}) \leqslant c_a(\vec{A})$ and of course an analogous argument using T^{-1} in place of T shows that $c_a(\vec{A}) \leqslant c_b(\vec{B})$. This finishes the proof of Fact 3.5.

To further study the action of linear maps on the quantities c_a, it is convenient to introduce the counterpart of the classical Banach–Mazur distance for Banach couples.

We have the following definition.

Definition 3.6. Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples. If A_j is isomorphic to B_j for $j = 0, 1$, then the ***Banach–Mazur distance*** between $\vec{A}$ and $\vec{B}$ is defined by

$$d(\vec{A}; \vec{B}) = \inf\big\{\|T\|_{\vec{A}\to\vec{B}} \big\|T^{-1}\big\|_{\vec{B}\to\vec{A}}\big\},$$

the infimum being taken over all linear isomorphisms $T : \vec{A} \to \vec{B}$. If there is no such isomorphism, we put $d(\vec{A}; \vec{B}) = \infty$.

Similarly, we define the distance between two elements $a \in A_0 + A_1$ and $b \in B_0 + B_1$ relative to $\vec{A}$ and $\vec{B}$ by

$$\operatorname{dist}_{\vec{A},\vec{B}}(a, b) = \inf_{Ta=b} \big\{\|T\|_{\vec{A}\to\vec{B}} \big\|T^{-1}\big\|_{\vec{B}\to\vec{A}}\big\}.$$

Let us return to Theorem 3.3 and consider a couple $\vec{X}$ of the form stated there. We use John's theorem to choose a two-dimensional Hilbert space Z_0 such that $\|\cdot\|_{Z_0} \leqslant \|\cdot\|_{X_0} \leqslant \sqrt{2}\|\cdot\|_{Z_0}$ and let $Z_1 = X_1$. The couple $\vec{Z}$ is then a Hilbert couple, necessarily isometric to a rigid image of the couple $\vec{Y}$. Thus we have $\gamma(\vec{Z}) = 2/\sqrt{3}$. The proof of Theorem 3.3 is now completed by the statement of the following proposition.

Proposition 3.7. *Let $\vec{A}$ and $\vec{B}$ be Gagliardo complete Banach couples. Fix $a \in A_0 + A_1$ and $b \in B_0 + B_1$. Then*

$$c_a(\vec{A}) \leqslant c_b(\vec{B}) \operatorname{dist}_{\vec{A},\vec{B}}(a, b). \tag{3.15}$$

In particular, $\gamma(\vec{A}) \leqslant \gamma(\vec{B}) d(\vec{A}; \vec{B})$.

Proof. Write $c_a = c_a(\vec{A})$ and $c_b = c_b(\vec{B})$. Assume that $C = \operatorname{dist}_{\vec{A},\vec{B}}(a, b)$ is finite. Take $\varepsilon > 0$ and let $T : \vec{A} \to \vec{B}$ be an isomorphism such that $Ta = b$ and $\|T\|\|T^{-1}\| < C + \varepsilon$. Let $\vec{P}$ and $\vec{Q}$ be weighted L^1-couples and $f \in P_0 + P_1$ and $\bar{f} \in Q_0 + Q_1$ be elements such that $K(t, f; \vec{P}) = K(t, a; \vec{A})$ and $K(t, \bar{f}; \vec{Q}) = K(t, b; \vec{B})$ for all $t > 0$. Then $K(t, \bar{f}; \vec{Q}) \leqslant \|T\| K(t, f; \vec{P})$ for all $t > 0$ and it follows from the Sedaev–Semenov theorem that there is a map $S : \vec{P} \to \vec{Q}$ such that $Sf = \bar{f}$ and

$\|S\| < \|T\| + \varepsilon$. Let $R : \vec{Q} \to \vec{B}$ be a map such that $R\bar{f} = b$ and $\|R\| < c_b + \varepsilon$. Put $Ug = T^{-1}RSg$ for $g \in P_0 + P_1$. Then $Uf = a$ and $c_a \leqslant \|U\| \leqslant Cc_b + \mathcal{O}(\varepsilon)$. This proves the estimate (3.15).

The last statement of the proposition follows by considering the suprema over a and b in (3.15). □

Remark 3.8. Let $\vec{H}$ be a finite-dimensional Hilbert couple. Then it is easy to see that there exists a finite sequence $\lambda = (\lambda_i)_{i=1}^n \subset [0, \infty]$ such that $\vec{H}$ is isometric to the weighted ℓ^2-couple $(\ell_n^2, \ell_n^2(\lambda))$. A generalization of this statement to the case of infinite-dimensional Hilbert couples has been given by Sedaev [22]. By this latter observation, the interpolation of Hilbert couples becomes essentially the same as that of weighted ℓ^2-couples. (Cf. also [17,2,1].)

3.3. Calderón constants for finite-dimensional couples

In this subsection, we prove estimates for the relative Calderón constants for couples of a given finite dimension. The definition of these constants, which generalizes the notion of the K-divisibility constant, is the following.

Definition 3.9. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and assume in the following that all Banach spaces are over the field $\mathbb{K}$.

Let C be a non-negative constant. Two couples $\vec{A}$, $\vec{B}$ are ***relative C-monotonic couples*** if for every $\varepsilon > 0$, all $\alpha \in A_0 + A_1$ and $\beta \in B_0 + B_1$ such that

$$K(t, \beta; \vec{B}) \leqslant K(t, \alpha; \vec{A}), \quad t > 0, \tag{3.16}$$

there exists a $\mathbb{K}$-linear operator $T = T_\varepsilon : \vec{A} \to \vec{B}$ such that

$$T\alpha = \beta \quad \text{and} \quad \|T\|_{\vec{A}\to\vec{B}} < C + \varepsilon.$$

The smallest constant C satisfying this implication is called the ***Calderón constant*** relative to $\vec{A}$ and $\vec{B}$ and is denoted by $c(\vec{A}; \vec{B})$. We also put

$$c_n(\mathbb{K}) = \sup\bigl\{c(\vec{A}; \vec{B}):\ \dim_{\mathbb{K}}(A_i) \leqslant n \text{ and } \dim_{\mathbb{K}}(B_i) \leqslant n,\ i = 0, 1\bigr\}.$$

Calderón constants for pairs of weighted L^p spaces were studied at length in [24].

As for the case of K-divisibility constant, it is advantageous to think of the Calderón constants as functions of Banach couples under the Banach–Mazur metric. We have the following lemma.

Lemma 3.10. *Let $\vec{A}^i$ and $\vec{B}^i$ be non-zero Banach couples for $i = 1, 2$. Then*

$$c\bigl(\vec{A}^1; \vec{B}^1\bigr) \leqslant d\bigl(\vec{A}^1; \vec{A}^2\bigr)c\bigl(\vec{A}^2; \vec{B}^2\bigr)d\bigl(\vec{B}^1; \vec{B}^2\bigr).$$

Proof. We may assume that both of the Banach–Mazur distances above are finite, because otherwise the statement is trivial.

Take $\alpha \in A_0^1 + A_1^1$ and $\beta \in B_0^1 + B_1^1$ such that $K(t, \beta; \vec{B}^1) \leqslant K(t, \alpha; \vec{A}^1)$ for all $t > 0$. Let $T_A : \vec{A}^1 \to \vec{A}^2$ and $T_B : \vec{B}^1 \to \vec{B}^2$ be isomorphisms such that $\|T_A\|\|T_A^{-1}\| < d(\vec{A}^1; \vec{A}^2) + \epsilon$ and $\|T_B\|\|T_B^{-1}\| < d(\vec{B}^1; \vec{B}^2) + \epsilon$. It follows that

$$\|T_B\|^{-1} K\bigl(t, T_B(\beta); \vec{B}^2\bigr) \leqslant K\bigl(t, \alpha; \vec{A}^1\bigr) \leqslant \bigl\|T_A^{-1}\bigr\| K\bigl(t, T_A(\alpha); \vec{A}^2\bigr)$$

for all $t > 0$. Take $\epsilon > 0$. It then follows that there exists an operator $T_0 : \vec{A}^2 \to \vec{B}^2$ such that $T_0(T_A(\alpha)) = T_B(\beta)$ of norm at most $(c(\vec{A}^2; \vec{B}^2) + \epsilon)\|T_B\|\|T_A^{-1}\|$. The operator $T : \vec{A}^1 \to \vec{B}^1$ defined by $T = T_B^{-1} T_0 T_A$ then fulfills $T(\alpha) = \beta$ and $\|T\| \leqslant \|T_B^{-1}\|(c(\vec{A}^2; \vec{B}^2) + \epsilon)\|T_B\|\|T_A^{-1}\|\|T_A\| < d(\vec{A}^1; \vec{A}^2)c(\vec{A}^2; \vec{B}^2)d(\vec{B}^1; \vec{B}^2) + \mathcal{O}(\epsilon)$. □

We have the following theorem. The result as well as the method of proof is closely related to that of [8, Section 3].

Theorem 3.11. *$c_n(\mathbb{C}) = n$ and $n/\sqrt{2} \leqslant c_n(\mathbb{R}) \leqslant n$ for all $n \in \mathbb{N}$.*

Remark 3.12. In [8], Brudnyi and Shteinberg introduce the quantity $\varkappa_n$ defined by

$$\varkappa_n = \sup\bigl\{c(\vec{A}; \vec{A}):\ \dim(A_i) \leqslant n \text{ for } i = 0, 1\bigr\},$$

where the supremum is taken with respect to Banach couples over the reals. In [8, Theorem 3.1], it is shown that $n/2\sqrt{2} \leqslant \varkappa_n \leqslant n\sqrt{2}$. Since of course $\varkappa_n \leqslant c_n(\mathbb{R})$, our result provides a somewhat better upper estimate for $\varkappa_n$.

Proof of Theorem 3.11. "$\leqslant$": Let $\vec{A}$ and $\vec{B}$ be couples such that all the spaces A_i and B_i are of dimension at most n (scalars can be real or complex). Let $\alpha \in A_0 + A_1$ and $\beta \in B_0 + B_1$ be elements satisfying (3.16). Use John's theorem to find Hilbert spaces H_i and K_i such that $d(\vec{A}; \vec{H}) \leqslant \sqrt{n}$ and $d(\vec{B}; \vec{K}) \leqslant \sqrt{n}$. By Lemma 3.10

$$c(\vec{A}; \vec{B}) \leqslant nc(\vec{H}; \vec{K}).$$

But Hilbert couples are exact relative Calderón couples, i.e., $c(\vec{H}; \vec{K}) \leqslant 1$ by Theorem 2.2 of [2]. Thus $c(\vec{A}; \vec{B}) \leqslant n$.

"$\geqslant$": First assume complex scalars and define the space $\ell_n^{p,r}(q)$ for suitable fixed values of p, q and r by the norm

$$\|x\|^p_{\ell_n^{p,r}(q)} = \sum_{k=1}^n \bigl|q^{-kr}x_k\bigr|^p, \quad x = (x_k)_1^n \in \mathbb{C}^n.$$

For fixed p and q we also define the couple $\vec{\ell}_n^p(q) = (\ell_n^{p,0}(q), \ell_n^{p,1}(q))$. (The usual conventions apply for the case $p = \infty$.)

Choose a fixed $q > 1$ and put $h = (\sqrt{q}, \sqrt{q}^2, \dots, \sqrt{q}^n) \in \mathbb{C}^n$. As is shown in [8], we have

$$K\bigl(t, h; \vec{\ell}_n^1(q)\bigr) = \sum_{k=1}^n q^{k/2} \min\bigl\{1, q^{-k}t\bigr\} \leqslant \frac{\sqrt{q}-1}{\sqrt{q}+1} K\bigl(t, h; \vec{\ell}_n^\infty(q)\bigr). \tag{3.17}$$

(It is convenient to first prove the inequality in the cases $t = q^i$, and then use the concavity of the K-functional.)

By (3.17) there exists an operator $T: \vec{\ell}_n^\infty(q) \to \vec{\ell}_n^1(q)$ such that $T(h) = h$ and $\|T\| \leqslant \frac{\sqrt{q}+1}{\sqrt{q}-1} c(\vec{\ell}_n^\infty(q); \vec{\ell}_n^1(q))$.

Since we are assuming complex scalars, the Riesz–Thorin theorem can be applied. It yields that

$$\|T\|_{\ell_n^{\infty,1/2}(q) \to \ell_n^{1,1/2}(q)} \leqslant \frac{\sqrt{q}+1}{\sqrt{q}-1} c\bigl(\vec{\ell}_n^\infty(q); \vec{\ell}_n^1(q)\bigr).$$

This in turn yields

$$n = \|h\|_{\ell_n^{1,1/2}(q)} \leqslant \frac{\sqrt{q}+1}{\sqrt{q}-1} c\bigl(\vec{\ell}_n^\infty(q); \vec{\ell}_n^1(q)\bigr) \|h\|_{\ell_n^{\infty,1/2}(q)} = \frac{\sqrt{q}+1}{\sqrt{q}-1} c\bigl(\vec{\ell}_n^\infty(q); \vec{\ell}_n^1(q)\bigr).$$

It follows that $c_n(\mathbb{C}) \geqslant c(\vec{\ell}_n^\infty(q); \vec{\ell}_n^1(q)) \geqslant n\frac{\sqrt{q}-1}{\sqrt{q}+1}$. Since q can be chosen arbitrarily large, this gives $c_n(\mathbb{C}) \geqslant n$. The modifications necessary to treat the real case are carried out as in [8]. $\square$

We end this subsection with an open question.

Question 2. Is $c_n(\mathbb{R}) = n$?

3.4. On the case of a regular two-dimensional Hilbert couple

Let r be a positive number and let $\vec{G} = (G_0, G_1)$ be the couple for which $G_0 = \ell_2^2$ and G_1 is the weighted version of ℓ_2^2 with norm $\|(x, y)\|_{G_1} = \sqrt{x^2 + ry^2}$.

In this subsection we will prove a rather simple estimate: $\gamma(\vec{G}) < \sqrt{2}$.

Let us remark first that in the trivial case where $r = 1$ we obtain $\gamma(\ell_2^2, \ell_2^2) = 1$. In the general case, Proposition 3.7 yields that $\gamma(\vec{G})$ is a continuous function of r and $\gamma(\vec{G}) \leqslant \max(\sqrt{r}, 1/\sqrt{r})$. (This is because the Banach–Mazur distance between $\vec{G}$ and (ℓ_2^2, ℓ_2^2) is $\max(\sqrt{r}, 1/\sqrt{r})$.)

Fix a point $\alpha = (b, c) = (\cos a, \sin a) \in G_0 + G_1$ where $a \in [0, 2\pi)$. In fact, by Remark 2.2, we only need to consider the case where $a \in [0, \pi/2]$.

We will look for a parametric representation of the curve which is the boundary $\partial\Gamma(\alpha)$ of the Gagliardo diagram of α.

First let us fix some $t > 0$ and determine the point $z = (x, y)$ for which the infimum $K_2(t, \alpha; G_0, G_1)^2 = \inf_{z \in \mathbb{R}^2} \|z\|^2_{G_0} + t\|\alpha - z\|^2_{G_1}$ is attained. The point which we are looking for is of course the unique critical point of the function $\phi(x, y) = x^2 + y^2 + t(x-b)^2 + tr(y-c)^2$, i.e. $x = \frac{tb}{1+t}$ and $y = \frac{trc}{1+tr}$.

It is clear that, for this choice of z, the point $(\|z\|^2_{G_0}, \|\alpha - z\|^2_{G_1})$ belongs to $\partial\Gamma(\alpha)$, and that, furthermore, as t ranges over $(0, \infty)$ we obtain all points of $\partial\Gamma(\alpha) \cap \{(x_0, x_1)\colon x_0 > 0,\ x_1 > 0\}$ in this way. We note that $b - x = \frac{b+tb-tb}{1+t} = \frac{b}{1+t}$ and $c - y = \frac{c+trc-trc}{1+tr} = \frac{c}{1+tr}$. It follows that

$$\partial\Gamma(\alpha) \cap \bigl\{(x_0, x_1)\colon x_0 > 0,\ x_1 > 0\bigr\} = \bigl\{\bigl(\gamma_0(t), \gamma_1(t)\bigr)\colon 0 < t < \infty\bigr\}, \tag{3.18}$$

where the functions γ_0 and γ_1 are given by

$$\gamma_0(t) = t\sqrt{\frac{b^2}{(1+t)^2} + \frac{r^2c^2}{(1+tr)^2}} \quad \text{and} \quad \gamma_1(t) = \sqrt{\frac{b^2}{(1+t)^2} + \frac{rc^2}{(1+tr)^2}} \quad \text{for all } t \in (0, \infty).$$

Obviously $\gamma_1(t)$ is a strictly decreasing function of t. Since $\gamma_0(1/t)^2 = \frac{b^2}{(t+1)^2} + \frac{r^2c^2}{(t+r)^2}$ it is also clear that $\gamma_0(t)$ is a strictly increasing function of t.

Considering the limits of γ_0 and γ_1 as t tends to 0 and to ∞, we deduce that $\partial\Gamma(\alpha)$ is the union of the curve specified in (3.18) with the two rays on the coordinate axes

$$\bigl\{(0, v)\colon \sqrt{b^2 + rc^2} \leqslant v < \infty\bigr\} \quad \text{and} \quad \bigl\{(v, 0)\colon 1 \leqslant v < \infty\bigr\}. \tag{3.19}$$

Next we define two functions w_0 and w_1 by $w_0(t) := \gamma_0'(t)$ and $w_1(t) := -\gamma_1'(t)$ for all $t \in (0,\infty)$. These will turn out to be convenient weight functions to use in a couple of weighted L^1 spaces on $(0,\infty)$ as an essential step for calculating $\gamma(\vec{G})$. We note that (3.19) implies

$$\int_0^\infty w_0(t)\,dt = 1 \quad \text{and} \quad \int_0^\infty w_1(t)\,dt = \sqrt{b^2 + rc^2}. \tag{3.20}$$

We will see that routine calculations show that w_0 and w_1 are given explicitly by

$$w_j(t) = \frac{\frac{b^2}{(1+t)^3} + \frac{r^2c^2}{(1+rt)^3}}{\sqrt{\frac{b^2}{(1+t)^2} + \frac{r^{2-j}c^2}{(1+rt)^2}}} \quad \text{for } j = 0, 1 \text{ and } t \in (0,\infty). \tag{3.21}$$

The proof of this in the case $j = 1$ is immediate. For the case $j = 0$ we can first observe that

$$\gamma_0'(1/t) \cdot \frac{1}{t^2} = -\frac{d}{dt}\big(\gamma_0(1/t)\big) = \frac{\frac{b^2}{(1+t)^3} + \frac{r^2c^2}{(t+r)^3}}{\sqrt{\frac{b^2}{(1+t)^2} + \frac{r^2c^2}{(t+r)^2}}}$$

which implies that

$$w_0(1/t) = \frac{\frac{t^3b^2}{(1+t)^3} + \frac{t^3r^2c^2}{(t+r)^3}}{t\sqrt{\frac{b^2}{(1+t)^2} + \frac{r^2c^2}{(t+r)^2}}} = \frac{\frac{b^2}{(1/t+1)^3} + \frac{r^2c^2}{(1+r/t)^3}}{\sqrt{\frac{b^2}{(1/t+1)^2} + \frac{r^2c^2}{(1+r/t)^2}}}$$

which immediately gives (3.21) for $j = 0$.

Note that w_0 and w_1 are both strictly positive on $(0,\infty)$.

We will use the couple $\vec{P} = (P_0, P_1)$ of weighted L^1 spaces on the measure space $(0,\infty)$ (equipped with Lebesgue measure) where $P_0 = L^1_{w_0}$ and $P_1 = L^1_{w_1}$. Let f be the function which equals 1 identically on $(0,\infty)$. We will show that

$$K(t, f; \vec{P}) = K(t, \alpha; \vec{G}) \quad \text{for all } t > 0. \tag{3.22}$$

For each $t > 0$ it is well known and very easy to check that

$$K(t, f; \vec{P}) = \int_0^\infty \min\big\{w_0(s), tw_1(s)\big\}\,ds \tag{3.23}$$

and that an optimal decomposition $f = f_{0,t} + f_{1,t}$, for which the infimum in the calculation of (3.23) is attained, is given by $f_{0,t} = \chi_{E_t}$ and $f_{1,t} = \chi_{(0,\infty)\setminus E_t}$, where

$$E_t = \big\{s > 0\colon\ w_0(s) < tw_1(s)\big\}. \tag{3.24}$$

We need to consider the function

$$\frac{w_0(t)^2}{w_1(t)^2} = \frac{\frac{b^2}{(1+t)^2} + \frac{rc^2}{(1+tr)^2}}{\frac{b^2}{(t+1)^2} + \frac{r^2c^2}{(1+rt)^2}} = 1 + \frac{(r - r^2)c^2}{b^2(r + \frac{1-r}{t+1})^2 + r^2c^2}. \tag{3.25}$$

In the trivial cases where (b, c) is either $(0, 1)$ or $(1, 0)$ this is a constant function, and it a simple matter to check that (3.22) holds. (In the first case the K-functionals on the left and right sides of (3.22) both equal $\min\{1, t\}$ and in the second case they both equal $\min\{1, t\sqrt{r}\}$.)

In the remaining non-trivial case when b and c are both non-zero it is easy to see from (3.25) that, for any $r \in (0,\infty)$ with $r \neq 1$,

$$\frac{w_0(t)}{w_1(t)} \text{ is a strictly increasing continuous function of } t \text{ on } (0,\infty). \tag{3.26}$$

(The two cases $r < 1$ and $r > 1$ have to be considered separately.)

We introduce and calculate two "limiting" values of t by setting

$$t_0^2 := \lim_{s\to 0} \frac{w_0(s)^2}{w_1(s)^2} = 1 + \frac{(r - r^2)c^2}{b^2 + r^2c^2} = \frac{b^2 + rc^2}{b^2 + r^2c^2} \tag{3.27}$$

and

$$t_\infty^2 := \lim_{s\to\infty} \frac{w_0(s)^2}{w_1(s)^2} = 1 + \frac{(r - r^2)c^2}{r^2(b^2 + c^2)} = b^2 + c^2/r. \tag{3.28}$$

The property (3.26) implies that the set E_t defined in (3.24) is an open interval of the form $E_t = (0, u(t))$, where u is a non-decreasing function of t. By (3.27) and (3.28) we see that $u(t) = 0$ for $t \leqslant t_0$ and $u(t) = \infty$ for $t \geqslant t_\infty$, and, for each $t \in (t_0, t_\infty)$, $u(t)$ is the unique number in $(0, \infty)$ for which $w_0(u(t))/w_1(u(t)) = t$.

We can now deduce that, for $t \in (t_0, t_\infty)$, $\|f_{0,t}\|_{P_0} = \int_0^{u(t)} w_0(s)\,ds = \int_0^{u(t)} \gamma_0'(s)\,ds = \gamma_0(u(t)) - \gamma_0(0) = \gamma_0(u(t))$ and $\|f_{1,t}\|_{P_1} = \int_{u(t)}^\infty w_1(s)\,ds = -\int_{u(t)}^\infty \gamma_1'(s)\,ds = \gamma_1(u(t)) - \lim_{r\to\infty}\gamma_1(r) = \gamma_1(u(t))$.

This shows that, as t ranges over the interval (t_0, t_∞), the point $(\|f_{0,t}\|_{P_0}, \|f_{1,t}\|_{P_1})$ ranges over the curve (3.18), i.e., $\Gamma(f) = \Gamma(\alpha)$. By the well-known relation between K-functionals and Gagliardo diagrams (see [5, Section 7.1]), this implies that (3.22) holds.

It is clear that every bounded operator $T : \vec{P} \to \vec{G}$ uniquely determines and is uniquely determined by a suitable pair of (equivalence classes of) measurable functions $g_j : (0, \infty) \to \mathbb{R}$ for $j = 0, 1$, via the formula

$$Th = \left(\int_0^\infty g_0(s)h(s)\,ds, \int_0^\infty g_1(s)h(s)\,ds \right) \quad \text{for all } h \in L^1_{w_0} + L^1_{w_1}. \tag{3.29}$$

When it is necessary to explicitly indicate the connection between the operator T and the functions g_0 and g_1 which define it via (3.29), we will use the notation T_{g_0,g_1} in place of T.

Of course we need to be more explicit about the conditions that the functions g_0 and g_1 must satisfy. Straightforward arguments (exactly like the proof below of the equivalence of conditions (4.9) and (4.11) using the Lebesgue differentiation theorem and a suitable form of Minkowski's or Schwarz' inequality) show that the norm of T is given by

$$\|T\|_{\vec{P}\to\vec{G}} = \max_{j=0,1} \left\{ \operatorname*{ess\,sup}_{(0,\infty)} \frac{\sqrt{g_0^2 + r^j g_1^2}}{w_j} \right\} \tag{3.30}$$

and so g_0 and g_1 must be such that this expression in finite.

Remark 3.13. For our purposes, we can without loss of generality assume that $r > 1$, since for each $r < 1$, the couple $\vec{G}$ is a rigid image of the corresponding couple where r has been replaced by $1/r$. (Use Fact 3.5 and the rigid map $(x, y) \mapsto (y/\sqrt{r}, x/\sqrt{r})$.)

Now we will consider the class $\mathcal{T} = \mathcal{T}_{b,c}$ of all bounded operators $T : \vec{P} \to \vec{G}$ which satisfy $Tf = \alpha$ and consider the quantity $c_a = c_a(\vec{G}) = \inf\{\|T\|: T \in \mathcal{T}\}$. We first make a simple observation:

Proposition 3.14. *We have $c_0 = c_{\pi/2} = 1$ and if $a \in (0, \pi/2)$ then $c_a < \sqrt{1+b^2}$. In particular, $\gamma(\vec{G}) < \sqrt{2}$.*

Proof. By Remark 3.13 we can and will assume that $r > 1$.

If $a = 0$, i.e., if $(b, c) = (1, 0)$, then we have that $w_0^2(t) = w_1^2(t) = \frac{1}{(1+t)^2}$ and the operator $T = T_{g_0,g_1}$ defined by $g_0(s) = \frac{1}{(1+s)^2}$ and $g_1(s) = 0$ satisfies $Tf = (b, c)$ and $\|T\| = 1$. Thus $c_0 = 1$. The proof of the fact that $c_{\pi/2} = 1$ is equally simple. It uses the functions $g_0(s) = 0$ and $g_1(s) = \frac{r}{(1+rs)^2}$.

Now let $a \in (0, \pi/2)$. We claim that it suffices to consider the operator $T = T_{g_0,g_1}$ given by $g_0(s) = bw_0(s)$ and $g_1(s) = cw_1(s)/\sqrt{b^2 + rc^2}$. Indeed $Tf = \alpha$ by (3.20), and furthermore, by (3.27),

$$\frac{g_0^2(s) + g_1^2(s)}{w_0^2(s)} = b^2 + \frac{c^2}{b^2 + rc^2}\frac{w_1(s)^2}{w_0(s)^2} \leqslant b^2 + c^2\frac{b^2 + r^2c^2}{(b^2 + rc^2)^2} < b^2 + 1.$$

Similarly, (3.28) yields the estimate

$$\frac{g_0^2(s) + rg_1^2(s)}{w_1^2(s)} = b^2\frac{w_0^2(s)}{w_1^2(s)} + \frac{rc^2}{b^2 + rc^2} \leqslant b^2\big(b^2 + c^2/r\big) + \frac{rc^2}{b^2 + rc^2} < b^2 + 1.$$

We conclude that $c_a < \sqrt{b^2 + 1}$. It follows that $\gamma(\vec{G}) < \sqrt{2}$ (the function $a \mapsto c_a$ is continuous by Proposition 3.7). □

Remark 3.15. The above proposition combined with a simple application of Proposition 3.7 and John's theorem, and also with Theorem 3.3, shows that $\gamma(\vec{X}) < 2$ for every two-dimensional (real) Banach couple $\vec{X}$.

3.4.1. Further discussion

From here onwards, in view of Remark 2.2, and since we have seen that $c_0 = c_{\pi/2} = 1$, we need only consider the case where $a \in (0, \pi/2)$ and so the numbers b and c are strictly positive.

We will also suppose that $r > 1$ (cf. Remark 3.13).

Let $T = T_{g_0,g_1}$ be a member of $\mathcal{T}_{b,c}$ for which the infimum

$$c_a = \inf_{T\in\mathcal{T}_{b,c}} \|T\|_{\vec{P}\to\vec{G}} \tag{3.31}$$

is attained. Lemma 2.5 guarantees that such an operator T exists.

The exact value of c_a evades us at this point, but we hope that the following remarks will provide a step on the way towards calculating c_a and therefore also $\gamma(\vec{G})$. We will show below that the functions g_0, g_1 possess certain properties. We will also prove the estimate $c_a < (1+\sqrt{r})/2$. This will imply, in view of Proposition 3.14, that

$$\gamma(\vec{G}) < \min\left\{\frac{1+\sqrt{r}}{2}, \sqrt{2}\right\}. \tag{3.32}$$

Let $\widetilde{g}_0$ and $\widetilde{g}_1$ be the functions defined by $\widetilde{g}_j := \frac{|\int_0^\infty g_j(s)\,ds|}{\int_0^\infty |g_j(s)|\,ds}|g_j|$ for $j=0,1$. It is easy to check that the operator $\widetilde{T} = T_{\widetilde{g}_0,\widetilde{g}_1}$ is also in $\mathcal{T}_{b,c}$ and that $\|T_{\widetilde{g}_0,\widetilde{g}_1}\|_{\vec{P}\to\vec{G}} \leqslant \|T_{g_0,g_1}\|_{\vec{P}\to\vec{G}}$.

Hence, we can and will assume that g_0 and g_1 are non-negative a.e. The conditions on T imply that

$$\begin{cases} g_0^2+g_1^2 \leqslant c_a^2 w_0^2 & \text{and also} \\ g_0^2+rg_1^2 \leqslant c_a^2 w_1^2 & \text{at almost every point of } (0,\infty). \end{cases} \tag{3.33}$$

We introduce two subsets E_0, E_1 of $(0,\infty)$ defined by

$$E_i = \left\{s\in(0,\infty):\ g_0(s)^2 + r^i g_1(s)^2 = c_a^2 w_i(s)^2\right\}, \quad i=0,1.$$

The following simple fact is true.

Fact 3.16. *The set $E_0\cup E_1$ contains almost every point of* $(0,\infty)$.

Proof. Suppose, on the contrary, that there exists a set $E\subset(0,\infty)$ of positive measure, such that $g_0^2+g_1^2 < c_a^2w_0^2$ and also $g_0^2+rg_1^2 < c_a^2w_1^2$ at every point of E. Then we can suppose, replacing E if necessary by a smaller subset also having positive measure, that, for some positive ϵ,

$$g_0^2+g_1^2 < (1-\epsilon)c_a^2w_0^2 \quad \text{and also} \quad g_0^2+rg_1^2 < (1-\epsilon)c_a^2w_1^2 \quad \text{at all points of } E. \tag{3.34}$$

For $j=0,1$ we define the function $\widetilde{g}_j = \sqrt{g_j^2+\phi}$ where

$$\phi = \epsilon c_a^2 \chi_E \min\left\{\frac{w_0^2}{2}, \frac{w_1^2}{1+r}\right\}. \tag{3.35}$$

It follows easily from (3.33)–(3.35) that, for $j=0,1$, we have

$$\widetilde{g}_0^2 + r^j\widetilde{g}_1^2 = g_0^2 + r^j g_1^2 + \left(1+r^j\right)\phi \leqslant c_a^2 w_j^2 \tag{3.36}$$

at every point of E and at almost every point of $(0,\infty)\backslash E$.

Since w_0 and w_1 are both strictly positive on $(0,\infty)$ and E has positive measure, it follows that

$$\widetilde{b} := \int_0^\infty \widetilde{g}_0(s)\,ds > b = \int_0^\infty g_0(s)\,ds \quad \text{and} \quad \widetilde{c} := \int_0^\infty \widetilde{g}_1(s)\,ds > \int_0^\infty g_1(s)\,ds = c \tag{3.37}$$

and so the operator S defined by $S = T_{v_0,v_1}$ where $v_0 = \frac{b}{\widetilde{b}}\widetilde{g}_0$ and $v_1 = \frac{c}{\widetilde{c}}\widetilde{g}_1$ satisfies $Sf=\alpha$. In view of (3.36), (3.37) and (3.30), its norm satisfies $\|S\|_{\vec{P}\to\vec{G}} \leqslant \max\{\frac{b}{\widetilde{b}}, \frac{c}{\widetilde{c}}\}c_a < c_a$. This contradicts the minimal property of c_a, i.e. (3.31), and so proves Fact 3.16. □

It is convenient to restate Fact 3.16 slightly differently as:

$$\text{for a.e. } s\in(0,\infty) \quad \text{the point } \left(g_0(s), g_1(s)\right) \in \partial Q_s,$$

where the sets Q_s are defined by

$$Q_s := \left\{(x,y):\ x\geqslant 0,\ y\geqslant 0,\ x^2+y^2\leqslant c_a^2w_0^2(s),\ x^2+ry^2\leqslant c_a^2w_1^2(s)\right\}.$$

The boundary of Q_s consists of a segment of the x-axis, a segment of the y-axis, and subsets of the quarter circle C_s of radius $c_aw_0(s)$ and of the quarter ellipse $\varGamma_s$ with semi-axes of lengths $c_aw_1(s)$ and $\frac{1}{\sqrt{r}}c_aw_1(s)$ in the directions of the x- and y-axes respectively.

Since $r > 1$ we see from (3.25) that

$$w_0(s) < w_1(s) \tag{3.38}$$

and so, on and slightly above the x-axis, the points of Γ_s lie strictly to the right of C_s. On the other hand, since we shall show that

$$w_0(s) > \frac{1}{\sqrt{r}} w_1(s), \tag{3.39}$$

it will follow that the points of C_s on and near the y-axis lie strictly above Γ_s. The sets C_s and Γ_s intersect at a single point $(x(s), y(s))$ whose exact coordinates will be calculated in a moment. In view of (3.38) and (3.39) we will be able to assert that, apart from parts of the x- and y-axes, the boundary of Q_s consists of the circular arc C_s^* of radius $c_a w_0(s)$ from $(c_a w_0(s), 0)$ to $(x(s), y(s))$ and the portion Γ_s^* of the quarter ellipse Γ_s from $(x(s), y(s))$ to $(0, \frac{1}{\sqrt{r}} c_a w_1(s))$.

Let us now prove (3.39). Using (3.26) and (3.27) we see that it suffices to show that $\frac{b^2+rc^2}{b^2+r^2c^2} > \frac{1}{r}$, which is clear, since $rb^2 > b^2$.

To obtain explicit expressions for $x(s)$ and $y(s)$ we simply solve the two equations

$$x(s)^2 + y(s)^2 = c_a^2 w_0(s)^2 \quad \text{and} \quad x(s)^2 + r y(s)^2 = c_a^2 w_1(s)^2 \tag{3.40}$$

which gives $y(s)^2 = \frac{c_a^2(w_1(s)^2 - w_0(s)^2)}{r-1}$ and then $x(s)^2 = \frac{c_a^2(r w_0(s)^2 - w_1(s)^2)}{r-1}$. From this we deduce that

$$x(s) = c_a w_j(s) \frac{b(1+rs)}{\sqrt{b^2(1+rs)^2 + r^{1+j}c^2(1+s)^2}}, \quad j = 0, 1, \tag{3.41}$$

and

$$y(s) = c_a w_j(s) \frac{c\sqrt{r}(1+s)}{\sqrt{b^2(1+s)^2 + r^{1+j}c^2(1+rs)^2}}, \quad j = 0, 1. \tag{3.42}$$

Remark 3.17. In addition to Fact 3.16 it is now plain that, for the optimal functions g_0 and g_1 we have

$$g_0(s) \geqslant x(s) \quad \text{and} \quad g_1(s) \leqslant y(s) \quad \text{on } E_0$$

and likewise

$$g_0(s) \leqslant x(s) \quad \text{and} \quad g_1(s) \geqslant y(s) \quad \text{on } E_1.$$

At first glance one might suspect that $E_0 = E_1 = (0, \infty)$, i.e., that $g_0(s) = x(s)$ and $g_1(s) = y(s)$. However, if this were the case, we would have that

$$\int_0^\infty \frac{x(s)}{bc_a}\, ds \geqslant \frac{1}{c_a} > \frac{1}{\sqrt{2}} \quad \text{and} \quad \int_0^\infty \frac{y(s)}{cc_a}\, ds \geqslant \frac{1}{c_a} > \frac{1}{\sqrt{2}},$$

where we have used Proposition 3.14. On the other hand, a numerical calculation making use of the explicit formula (3.41) with the values $r = 1000$, $b = \sqrt{3}/2$ and $c = 1/2$ yields $\int_0^\infty (x(s)/c_a b)\, ds \approx 0.6896 < 1/\sqrt{2}$. Thus the functions x and y are not optimal in general.

We shall now use the operators $T = T_{x/c_a, y/c_a}$ to obtain some new information about $\gamma(\vec{G})$. From (3.40) and (3.30) it is evident that $\|T\|_{\vec{P} \to \vec{G}} = 1$. In order to prove the estimate $c_a < (1+\sqrt{r})/2$ it clearly suffices to prove that $(Tf)_1 > \frac{2b}{1+\sqrt{r}}$ and $(Tf)_2 > \frac{2c}{1+\sqrt{r}}$, i.e.,

$$\int_0^\infty \frac{x(s)}{bc_a}\, ds > \frac{2}{1+\sqrt{r}} \quad \text{and} \quad \int_0^\infty \frac{y(s)}{cc_a}\, ds > \frac{2}{1+\sqrt{r}}. \tag{3.43}$$

In order to prove (3.43), we observe that, for $j = 0, 1$, the functions

$$u_j(s) := \frac{1+rs}{\sqrt{b^2(1+rs)^2 + r^{1+j}c^2(1+s)^2}} = 1\Big/\sqrt{b^2 + r^{1+j}c^2 \frac{(1+s)^2}{(1+rs)^2}}$$

are increasing on $(0, \infty)$ and, likewise, the functions

$$v_j(s) := \frac{\sqrt{r}(1+s)}{\sqrt{b^2(1+s)^2 + r^{1+j}c^2(1+rs)^2}}$$

are decreasing on $(0,\infty)$. By (3.41) we obtain

$$\int_0^\infty \frac{x(s)}{bc_a}\,ds = \int_0^{1/\sqrt{r}} u_0(s)\,d\gamma_0(s) + \int_{1/\sqrt{r}}^\infty u_0(s)\,d\gamma_0(s)$$
$$> u_0(0)\big(\gamma_0(1/\sqrt{r}) - \gamma_0(0)\big) + u_0(1/\sqrt{r})\big(1-\gamma_0(1/\sqrt{r})\big)$$
$$= \frac{1}{\sqrt{b^2+rc^2}}\cdot\frac{\sqrt{b^2+rc^2}}{1+\sqrt{r}} + 1\cdot\left(1-\frac{\sqrt{b^2+rc^2}}{1+\sqrt{r}}\right)$$
$$= 1+\frac{1-\sqrt{b^2+rc^2}}{1+\sqrt{r}} \geqslant 1+\frac{1-\sqrt{r}}{1+\sqrt{r}} = \frac{2}{1+\sqrt{r}}.$$

Similarly, by using (3.42), we get

$$\int_0^\infty \frac{y(s)}{cc_a}\,ds > v_1(1/\sqrt{r})\big(\gamma_1(0)-\gamma_1(1/\sqrt{r})\big) + v_1(\infty)\gamma_1(1/\sqrt{r})$$
$$= \frac{1}{\sqrt{b^2+rc^2}}\cdot\left(\sqrt{b^2+rc^2}-\frac{\sqrt{r}}{1+\sqrt{r}}\right) + \frac{\sqrt{r}}{1+\sqrt{r}}\cdot\frac{1}{\sqrt{r}}$$
$$\geqslant 1+\frac{\sqrt{r}}{1+\sqrt{r}}\left(\frac{1}{\sqrt{r}}-1\right) = 1+\frac{1-\sqrt{r}}{1+\sqrt{r}} = \frac{2}{1+\sqrt{r}}.$$

This establishes (3.43) and so indeed we have $c_a < (1+\sqrt{r})/2$ and can deduce (3.32).

4. The two-dimensional couple $\vec{X} = (\ell_2^2, \ell_2^\infty)$

4.1. Terminology, notation and some preliminaries

In this section we will study the couple $(\ell_2^2, \ell_2^\infty)$ which we will always denote by $\vec{X}$ or (X_0, X_1). We have seen that the couple $\vec{Y} = (\ell_2^2, \ell_1^2)$ is an exact Calderón couple (see [2]) which is *not exactly K-divisible*, i.e., for which $\gamma(\vec{Y}) > 1$. In this section, we shall see that $\vec{X}$ is another example of a couple having both these properties.

Lemma 4.1. *$\vec{X}$ is an exact Calderón couple.*

Proof. Suppose that $f = (f_0, f_1)$ and $g = (g_0, g_1)$ are two points in $\mathbb{R}^2$ which satisfy $K(t, g; \vec{X}) \leqslant K(t, f; \vec{X})$ for all $t > 0$. We will show that there exists an operator $T : \vec{X} \to \vec{X}$ with norm $\|T\|_{\vec{X}\to\vec{X}} \leqslant 1$ such that $Tf = g$. We can of course assume without loss of generality that $f_0 \geqslant f_1 \geqslant 0$ and $g_0 \geqslant g_1 \geqslant 0$. The K-functional inequality satisfied by f and g is equivalent to an E-functional inequality which can be written as

$$\big(f_0 - \min(t, f_0)\big)^2 + \big(f_1 - \min(t, f_1)\big)^2 \geqslant \big(g_0 - \min(t, g_0)\big)^2 + \big(g_1 - \min(t, g_1)\big)^2$$

and which holds for all $t > 0$.

It is clear that $f_0 \geqslant g_0$. (Otherwise we get a contradiction by choosing $t = (f_0 + g_0)/2$.) By setting $t = 0$ we also have that $f_0^2 + f_1^2 \geqslant g_0^2 + g_1^2$. This means that the condition

$$\int_0^t f^*(s)^2\,ds \geqslant \int_0^t g^*(s)^2\,ds \tag{4.1}$$

holds for $t = 0$, 1 and for all $t \geqslant 2$. (Here, our spaces ℓ_2^2 and ℓ_2^∞ coincide with L^2 and L^∞ on a measure space consisting of two atoms each of measure 1. So the rearrangements of f and g are $f^* = f_0\chi_{[0,1)} + f_1\chi_{[1,2)}$ and $g^* = g_0\chi_{[0,1)} + g_1\chi_{[1,2)}$.)

Since both sides of (4.1) are affine functions on $[0,1]$ and $[1,2]$ it follows that (4.1) holds for all $t > 0$. Then we can apply the theorem and proof of Lorentz and Shimogaki [19] to construct the required operator T. □

Consider the point $\alpha = (a, 1) \in X_0 + X_1$ where $a > 1$. Let $E(t, \alpha; \vec{X})$ be the error functional

$$E(t, \alpha; \vec{X}) = \inf\big\{\|\alpha - \beta\|_{X_0}\colon\ \beta \in X_1,\ \|\beta\|_{X_1} \leqslant t\big\}.$$

For $t \in (0,1]$, the optimal choice of β is (t,t), and for $t \in [1,a]$ the optimal choice of β is $(t,1)$, and for $t > a$ the optimal choice is $\beta = \alpha$. Consequently

$$E(t,\alpha;\vec{X}) = \begin{cases} \sqrt{(a-t)^2+(1-t)^2}, & 0 \leqslant t \leqslant 1, \\ a-t, & 1 < t \leqslant a, \\ 0, & t > a. \end{cases}$$

Now let $w : (0,a) \to (1,\infty)$ be a non-increasing function and consider the couple of weighted L^1 spaces $\vec{P} = (P_0, P_1)$ on the measure space $(0,a)$ (equipped with Lebesgue measure) where $P_0 = L^1_w$ and $P_1 = L^1$. Let $f = \chi_{(0,a)}$, and let $E(t,f;\vec{P}) = \inf\{\|f-g\|_{P_0}\colon\ g \in P_0,\ \|g\|_{P_1} \leqslant t\}$. Since $w \geqslant 1$ and w is non-increasing, the optimal choice for g is $\chi_{[0,\min(t,a)]}$ for all $t \in (0,\infty)$. It follows that $E(t,f;\vec{P}) = \|f-g\|_{P_0} = \int_{\min(t,a)}^a w(\xi)\,d\xi$.

If w is continuous, then $E(t,f;\vec{P})$ is differentiable, with derivative equal to $-w(t)$ for all $t \in (0,a)$.

The function $E(t,\alpha;\vec{X})$ is also differentiable on $(0,a)$ and its derivative for $t \in (0,a)$ is given by

$$\frac{d}{dt}E(t,\alpha;\vec{X}) = \begin{cases} \frac{2t-a-1}{\sqrt{(a-t)^2+(1-t)^2}}, & 0 < t < 1, \\ -1, & 1 \leqslant t < a. \end{cases}$$

By general properties of the error functional, this derivative must be negative and non-decreasing. Thus the function

$$w_*(t) := -\frac{d}{dt}E(t,\alpha;\vec{X}) = \begin{cases} \frac{a+1-2t}{\sqrt{(t-a)^2+(t-1)^2}}, & 0 < t < 1, \\ 1, & 1 \leqslant t < a \end{cases} \tag{4.2}$$

is continuous and non-increasing and $w_*(t) \geqslant 1$ on $(0,a)$. In fact, as can be shown directly, it is strictly decreasing on $(0,1]$. If we now choose $w = w_*$ then it is easy to check that $E(t,f;\vec{P}) = E(t,\alpha;\vec{X})$ for all $t > 0$. This is equivalent, using well-known connections between error functionals, K-functionals and the Gagliardo diagram, to the condition

$$K(t,f;\vec{P}) = K(t,\alpha;\vec{X}) \quad \text{for all } t > 0. \tag{4.3}$$

For the rest of this section w will always denote the particular function defined by (4.2), for some choice of the constant a. It is easy to check that, for every choice of $a > 1$, we have

$$1 \leqslant w(t) < \sqrt{2}, \quad \text{and so also} \quad \sqrt{w^2(t)-1} < 1, \quad \text{for all } t \in (0,a). \tag{4.4}$$

For each fixed $a \geqslant 1$, let $\mathcal{T}_a$ be the set of all bounded linear operators $T : \vec{P} \to \vec{X}$, which, for $f = \chi_{(0,a)}$ and $\alpha = (a,1)$ and w as above, satisfy $Tf = \alpha$.

Let T be an arbitrary operator in $\mathcal{T}_a$. Then T has the form

$$Th = \big(\lambda_0(h), \lambda_1(h)\big) \quad \text{for all } h \in P_0 + P_1,$$

where λ_0 and λ_1 are both elements of $(P_0)^* \cap (P_1)^*$ such that

$$\lambda_0(\chi_{(0,a)}) = a \quad \text{and} \quad \lambda_1(\chi_{(0,a)}) = 1.$$

The norm of T satisfies $\|T\|_{\vec{P}\to\vec{X}} \leqslant c$ for some positive constant c, if and only if

$$\big|\lambda_0(h)\big|^2 + \big|\lambda_1(h)\big|^2 \leqslant c^2\|h\|^2_{P_0} \quad \text{for all } h \in P_0$$

and

$$\|\lambda_j\|_{(P_1)^*} \leqslant c \quad \text{for } j = 0, 1.$$

We are interested in the quantity

$$c_a := \inf\big\{\|T\|_{\vec{P}\to\vec{X}}\colon\ T \in \mathcal{T}_a\big\}. \tag{4.5}$$

By (4.3) and standard properties of the K-functional we clearly have that

$$c_a \geqslant 1. \tag{4.6}$$

By Lemma 2.5 the infimum in (4.5) is attained for some $T \in \mathcal{T}_a$.

There is of course a more concrete version of the representation given above for operators $T \in \mathcal{T}_a$:

A bounded linear operator $T : \vec{P} \to \vec{X}$ is determined by two functions g_0 and g_1 in $L^\infty(0,a)$. We denote this operator by $T = T_{g_0,g_1}$, where

$$Th = T_{g_0,g_1}h = \left(\int_0^a h(\xi)g_0(\xi)\,d\xi,\ \int_0^a h(\xi)g_1(\xi)\,d\xi\right) \quad \text{for each } h \in P_0 + P_1. \tag{4.7}$$

Such an operator T_{g_0,g_1} is in $\mathcal{T}_a$ if and only if the functions g_0 and g_1 also satisfy

$$\int_0^a g_0(\xi)\,d\xi = a \quad \text{and} \quad \int_0^a g_1(\xi)\,d\xi = 1. \tag{4.8}$$

For any $T_{g_0,g_1} : \vec{P} \to \vec{X}$, the norm estimate $\|T_{g_0,g_1}\|_{\vec{P}\to\vec{X}} \leqslant c$ is equivalent to the two conditions

$$\left(\int_0^a h(\xi)g_0(\xi)\,d\xi\right)^2 + \left(\int_0^a h(\xi)g_1(\xi)\,d\xi\right)^2 \leqslant c^2\left(\int_0^a \left|h(\xi)\right|w(\xi)\,d\xi\right)^2 \quad \text{for all } h \in P_0 \tag{4.9}$$

and

$$\|g_j\|_{L^\infty} \leqslant c \quad \text{for } j = 0, 1. \tag{4.10}$$

In fact (4.9) is equivalent to

$$g_0(\xi)^2 + g_1(\xi)^2 \leqslant c^2 w^2(\xi) \quad \text{for a.e. } \xi \in (0, a). \tag{4.11}$$

The proof that (4.9) implies (4.11) follows readily from the Lebesgue differentiation theorem. To prove the reverse implication, we note that the square root of the left side of (4.9) is equal to

$$\sup_{\theta\in[0,2\pi]} \int_0^a h(\xi)g_0(\xi)\cos\theta + h(\xi)g_1(\xi)\sin\theta\,d\xi \tag{4.12}$$

and, by Schwarz' inequality, the absolute value of the integrand in (4.12) is dominated by $|h(\xi)|\sqrt{g_0(\xi)^2 + g_1(\xi)^2}$.

4.2. A simple estimate from below for $\gamma(\ell^2_2, \ell^\infty_2)$

We can now easily show that $\vec{X} = (\ell^2_2, \ell^\infty_2)$ is an example, perhaps the simplest known example so far, of a Banach couple whose K-divisibility constant satisfies

$$\gamma(\vec{X}) > 1. \tag{4.13}$$

Remark 4.2. This example is of interest for a number of reasons.

- It is apparently the first known example of a couple of rearrangement invariant spaces which is not exactly K-divisible.
- It also shows that there is no "tight" connection between the exact K-divisibility property and the exact Calderón property, since there are also exactly K-divisible couples which are not exact Calderón couples, or not even Calderón couples. An example is provided by the couple $(L^1 \oplus L^\infty, L^\infty \oplus L^1)$.

Using well-known results concerning K-divisibility (Theorem 2.3) it is easy to see that

$$\gamma(\vec{X}) = \sup_{a\geqslant 1} c_a$$

where c_a is defined by (4.5). We shall show that $c_a > 1$ for every $a > 1$.

Suppose, on the contrary, that $c_a = 1$ for some $a > 1$. (Recall (4.6).) Let T be the operator in $\mathcal{T}_a$ whose existence we established above, which satisfies $\|T\|_{\vec{P}\to\vec{X}} = c_a = 1$. Then there exist functions g_0 and g_1 in $L^\infty(0, a)$ satisfying (4.8) and also satisfying the estimates (4.10) and (4.11) for $c = 1$. In particular, since $\int_0^a g_0(\xi)\,d\xi = a$ and $|g_0(\xi)| \leqslant 1$ for a.e. $\xi \in (0, a)$, we must have $g_0(\xi) = 1$ a.e. It follows that

$$1 = \int_0^a g_1(\xi)\,d\xi \leqslant \int_0^a \sqrt{w^2(\xi) - 1}\,d\xi = \int_0^1 \sqrt{w^2(\xi) - 1}\,d\xi = \int_0^1 \sqrt{\frac{(2\xi - a - 1)^2}{(a-\xi)^2 + (1-\xi)^2} - 1}\,d\xi. \tag{4.14}$$

The expression under the square root in the last integral can be rewritten as

$$\frac{(a+1-2\xi)^2 - (a-\xi)^2 - (1-\xi)^2}{(a-\xi)^2 + (1-\xi)^2} = \frac{((a-\xi) + (1-\xi))^2 - (a-\xi)^2 - (1-\xi)^2}{(a-\xi)^2 + (1-\xi)^2}$$
$$= \frac{2(a-\xi)(1-\xi)}{(a-\xi)^2 + (1-\xi)^2}. \tag{4.15}$$

This equals 1 for all ξ if $a = 1$. But, for all $a > 1$, we have $\frac{2(a-\xi)(1-\xi)}{(a-\xi)^2+(1-\xi)^2} < 1$ for all ξ. This shows that (4.14) cannot hold, and so provides the contradiction which proves that $c_a > 1$ and also establishes (4.13).

Remark 4.3. It is easy to show that $c_a = 1$ when $a = 1$. In this case the function w assumes the constant value $\sqrt{2}$ on $(0, a) = (0, 1)$ and the operator $T = T_{g_0,g_1}$, which is obtained by simply choosing g_0 and g_1 to be both identically 1, is in $\mathcal{T}_a$ and satisfies $\|T\|_{\vec{P}\to\vec{X}} = 1$.

4.3. A more elaborate calculation

From here onwards a will always denote a fixed number satisfying $a > 1$, and g_0 and g_1 will always denote two particular functions in $L^\infty(0, a)$ which satisfy (4.7) and (4.8) for an operator $T_{g_0,g_1} \in \mathcal{T}_a$ which attains the infimum c_a in (4.5). Therefore g_0 and g_1 satisfy (4.10) and (4.11) with $c = c_a$. Our goal here will be to show that g_0 and g_1 necessarily have certain properties. Our calculations in this subsection will also lead to the estimate $\gamma(\vec{X}) \leqslant \frac{4+3\sqrt{2}}{4+2\sqrt{2}}$.

By familiar arguments we can and will assume that g_0 and g_1 are both non-negative.

We will use the following very simple claim several times in subsequent steps of our argument:

Claim 4.4. *Suppose that $\widetilde{g}_0$ and $\widetilde{g}_1$ are two non-negative functions in $L^\infty(0, a)$ which satisfy*

$$\int_0^a \widetilde{g}_0(\xi)\,d\xi > a \quad \text{and} \quad \int_0^a \widetilde{g}_1(\xi)\,d\xi > 1. \tag{4.16}$$

Then

$$\|T_{\widetilde{g}_0,\widetilde{g}_1}\|_{\vec{P}\to\vec{X}} > c_a.$$

Proof. Suppose, on the contrary that

$$\|T_{\widetilde{g}_0,\widetilde{g}_1}\|_{\vec{P}\to\vec{X}} \leqslant c_a. \tag{4.17}$$

Then the operator S defined by

$$Sh = \left(\frac{a}{\int_0^a \widetilde{g}_0(\xi)\,d\xi}\int_0^a \widetilde{g}_0(\xi)h(\xi)\,d\xi,\ \frac{1}{\int_0^a \widetilde{g}_1(\xi)\,d\xi}\int_0^a \widetilde{g}_1(\xi)h(\xi)\,d\xi\right)$$

has norm $\|S\|_{\vec{P}\to\vec{X}}$ strictly smaller than c_a. But $S \in \mathcal{T}_a$ and so we have a contradiction, which proves the claim. □

It will be convenient to define the planar set

$$E_\xi = \left\{(x, y) \in \mathbb{R}^2\colon\ 0 \leqslant x \leqslant c_a,\ 0 \leqslant y \leqslant c_a,\ x^2 + y^2 \leqslant c_a^2 w^2(\xi)\right\}$$

for each $\xi \in (0, a)$. Then, reformulating our remarks above, for any non-negative measurable functions u_0 and u_1 on $(0, a)$, $\|T_{u_0,u_1}\|_{\vec{P}\to\vec{X}} \leqslant c_a$ if and only if $(u_0(\xi), u_1(\xi)) \in E_\xi$ for a.e. $\xi \in (0, a)$. In particular, the two particular norm minimizing functions g_0 and g_1 which we are studying, satisfy this condition.

We note that the boundary of E_ξ consists of two horizontal and two vertical line segments and a circular arc of radius $c_a w(\xi)$ which we will denote by Γ_ξ. We let V_ξ denote the vertical segment of the right side of the boundary of E_ξ, i.e.

$$V_\xi = \left\{(c_a, y)\colon\ 0 \leqslant y \leqslant c_a\sqrt{w^2(\xi) - 1}\right\}.$$

The uppermost point of V_ξ, which is also the lowest point of Γ_ξ, is

$$\left(c_a, c_a\sqrt{w^2(\xi) - 1}\right) = \left(c_a w(\xi)\cos\psi(\xi), c_a w(\xi)\sin\psi(\xi)\right)$$
$$\text{where } \psi(\xi) = \arctan\sqrt{w^2(\xi) - 1} = \arccos\frac{1}{w(\xi)}. \tag{4.18}$$

Let $\mathcal{U}_a$ be the family of all couples (u_0, u_1) of non-negative functions in $L^\infty(0, a)$ which satisfy

(i) $(u_0(\xi), u_1(\xi)) \neq (0, 0)$ for a.e. $\xi \in (0, a)$, and
(ii) $\|T_{u_0,u_1}\|_{\vec{P}\to\vec{X}} \leqslant c_a$ or, equivalently $(u_0(\xi), u_1(\xi)) \in E_\xi$ for a.e. $\xi \in (0, a)$.

We claim that the special functions g_0 and g_1 satisfy

$$(g_0, g_1) \in \mathcal{U}_a. \tag{4.19}$$

They of course satisfy part (ii) of the definition. To show that they also satisfy part (i), let

$$N = \left\{\xi \in (0, a)\colon\ \left(g_0(\xi), g_1(\xi)\right) = (0, 0)\right\}$$

and let $\widetilde{g}_j = g_j \chi_{(0,a)\setminus N} + \frac{c_a}{\sqrt{2}} w \chi_N$ for $j = 0, 1$. In view of (4.4) it is clear that $(\widetilde{g}_0(\xi), \widetilde{g}_1(\xi)) \in E_\xi$ for a.e. $\xi \in (0,a)$, which is equivalent to (4.17). But, if N has positive measure, then (4.16) also holds, which, by Claim 4.4, is impossible.

It is convenient to represent each $(u_0, u_1) \in \mathcal{U}_a$ in the "polar" form $(u_0, u_1) = (\rho\cos\theta, \rho\sin\theta)$ where $\rho : (0,a) \to (0, \sqrt{2})$ and $\theta : (0,a) \to [0, \frac{\pi}{2}]$ are the measurable functions defined by $\rho(\xi) = \sqrt{u_0^2(\xi) + u_1^2(\xi)}$ and $\theta(\xi) = \arcsin \frac{u_1(\xi)}{\rho(\xi)}$ for all $\xi \in (0,a)$. Accordingly, we let $\mathcal{P}_a$ be the family of all couples (ρ, θ) of functions $\rho : (0,a) \to [0, \sqrt{2})$ and $\theta : (0,a) \to [0, \frac{\pi}{2}]$ such that $(\rho\cos\theta, \rho\sin\theta) \in \mathcal{U}_a$.

Claim 4.5. *If $(\rho, \theta) \in \mathcal{P}_a$ and $\phi : (0,a) \to [0, \frac{\pi}{2}]$ is a measurable function satisfying*

$$\theta(\xi) \leqslant \phi(\theta) \leqslant \frac{\pi}{4} \quad \text{or} \quad \theta(\xi) \geqslant \phi(\theta) \geqslant \frac{\pi}{4}$$

for a.e. $\xi \in (0,a)$, then $(\rho, \phi) \in \mathcal{P}_a$.

This is obvious, in view of the form of the sets E_ξ.

We have now the following simple "variational principle":

Lemma 4.6. *Suppose that the functions ρ and θ satisfy*

$$(\rho, \theta) \in \mathcal{P}_a \quad \text{and} \quad g_0 = \rho\cos\theta \quad \text{and} \quad g_1 = \rho\sin\theta. \tag{4.20}$$

Suppose that A and B are each measurable subsets of $(0,a)$ with positive measure. Suppose that p, q are real constants such that, for some $\delta > 0$ and each constant $t \in [0, \delta]$, the function $\phi_t = \theta + tp\chi_A + tq\chi_B$ satisfies

$$(\rho, \phi_t) \in \mathcal{P}_a. \tag{4.21}$$

Then at least one of the following two inequalities

$$p\int_A \rho(\xi)\sin\theta(\xi)\,d\xi + q\int_B \rho(\xi)\sin\theta(\xi)\,d\xi \geqslant 0 \tag{4.22}$$

and

$$p\int_A \rho(\xi)\cos\theta(\xi)\,d\xi + q\int_B \rho(\xi)\cos\theta(\xi)\,d\xi \leqslant 0 \tag{4.23}$$

must hold.

Proof. Define $G_0(t) = \int_0^a \rho(\xi)\cos\phi_t(\xi)\,d\xi$ and $G_1(t) = \int_0^a \rho(\xi)\sin\phi_t(\xi)\,d\xi$ for all $t \in \mathbb{R}$. Standard arguments (e.g. via dominated convergence) show that G_0 and G_1 are differentiable for all $t \in \mathbb{R}$ and their derivatives are continuous functions of t given by

$$G_0'(t) = -p\int_A \rho(\xi)\sin\phi_t(\xi)\,d\xi - q\int_B \rho(\xi)\sin\phi_t(\xi)\,d\xi$$

and

$$G_1'(t) = p\int_A \rho(\xi)\cos\phi_t(\xi)\,d\xi + q\int_B \rho(\xi)\cos\phi_t(\xi)\,d\xi.$$

Suppose that neither of (4.22) and (4.23) hold. Then $G_0'(0)$ and $G_1'(0)$ are both strictly positive. Thus G_0 and G_1 are both increasing functions in some neighborhood of 0. So, for some $\delta' \in (0, \delta]$, we have $G_0(\delta') > G_0(0)$ and $G_1(\delta') > G_1(0)$, or, in other words, the functions $\widetilde{g}_0 := \rho\cos\phi_{\delta'}$ and $\widetilde{g}_1 := \rho\sin\phi_{\delta'}$ satisfy (4.16). But, in view of (4.21), these same two functions also satisfy (4.17). By Claim 4.4 this is impossible, so at least one of (4.22) and (4.23) must hold. □

As our first application of Lemma 4.6 we will prove that

$$g_0(\xi) \geqslant g_1(\xi) \quad \text{for a.e. } \xi \in (0,a). \tag{4.24}$$

If the functions ρ and θ satisfy (4.20) then (4.24) is equivalent to

$$\theta(\xi) \leqslant \frac{\pi}{4} \quad \text{for a.e. } \in (0,a). \tag{4.25}$$

So, if (4.24) is false, then the set $\{\xi \in (0,a){:}\ g_0(\xi) < g_1(\xi)\} = \{\xi \in (0,a){:}\ \theta(\xi) > \frac{\pi}{4}\}$ has positive measure and, furthermore, for some positive number η_0, the set $A := \{\xi \in (0,a){:}\ \theta(\xi) > \eta_0 + \frac{\pi}{4}\}$ also has positive measure. Since

$\int_0^a g_0(\xi)\,d\xi > \int_0^a g_1(\xi)\,d\xi$ the set $\{\xi \in (0,a)\colon\ g_0(\xi) > g_1(\xi)\} = \{\xi \in (0,a)\colon\ \theta(\xi) < \frac{\pi}{4}\}$ must also have positive measure, and so, for some positive number η_1, the set $B = \{\xi \in (0,a)\colon\ \theta(\xi) < \frac{\pi}{4} - \eta_1\}$ also has positive measure. Let p be an arbitrary negative number and let $q = 1$. Let us also choose $\delta = \min\{\eta_0/|p|, \eta_1\}$. Then, using Claim 4.5, we see that all the hypotheses of Lemma 4.6 hold. Consequently, Lemma 4.6 implies that

$$p\int_A \rho(\xi)\sin\theta(\xi)\,d\xi + \int_B \rho(\xi)\sin\theta(\xi)\,d\xi \geqslant 0 \quad \text{or}$$

$$p\int_A \rho(\xi)\cos\theta(\xi)\,d\xi + \int_B \rho(\xi)\cos\theta(\xi)\,d\xi \leqslant 0.$$

But now we shall show that we have a contradiction by finding a negative number p which satisfies

$$\begin{cases} p\int_A \rho(\xi)\sin\theta(\xi)\,d\xi + \int_B \rho(\xi)\sin\theta(\xi)\,d\xi < 0 \quad \text{and} \\ p\int_A \rho(\xi)\cos\theta(\xi)\,d\xi + \int_B \rho(\xi)\cos\theta(\xi)\,d\xi > 0. \end{cases} \tag{4.26}$$

In view of (4.19), $\rho(\xi) > 0$ for a.e. ξ and $\int_A \rho(\xi)\sin\theta(\xi)\,d\xi > \int_A \rho(\xi)\sin\frac{\pi}{4}\,d\xi > 0$. We also have $\int_A \rho(\xi)\sin\theta(\xi)\,d\xi > \int_A \rho(\xi)\cos\theta(\xi)\,d\xi \geqslant 0$. Similarly $\int_B \rho(\xi)\cos\theta(\xi)\,d\xi > \int_B \rho(\xi)\cos\frac{\pi}{4}\,d\xi > 0$ and $\int_B \rho(\xi)\cos\theta(\xi)\,d\xi > \int_B \rho(\xi)\sin\theta(\xi)\,d\xi \geqslant 0$.

If $\int_A \rho(\xi)\cos\theta(\xi)\,d\xi = 0$ then every number $p < -\frac{\int_B \rho(\xi)\sin\theta(\xi)\,d\xi}{\int_A \rho(\xi)\sin\theta(\xi)\,d\xi}$ satisfies (4.26). Otherwise, if $\int_A \rho(\xi)\cos\theta(\xi)\,d\xi \neq 0$, then condition (4.26) is equivalent to

$$p + \frac{\int_B \rho(\xi)\sin\theta(\xi)\,d\xi}{\int_A \rho(\xi)\sin\theta(\xi)\,d\xi} < 0 \quad \text{and} \quad p + \frac{\int_B \rho(\xi)\cos\theta(\xi)\,d\xi}{\int_A \rho(\xi)\cos\theta(\xi)\,d\xi} > 0$$

and so also to

$$\frac{\int_B \rho(\xi)\sin\theta(\xi)\,d\xi}{\int_A \rho(\xi)\sin\theta(\xi)\,d\xi} < -p < \frac{\int_B \rho(\xi)\cos\theta(\xi)\,d\xi}{\int_A \rho(\xi)\cos\theta(\xi)\,d\xi}.$$

So it is clear that we can find p with the required properties, if and only if

$$\frac{\int_B \rho(\xi)\sin\theta(\xi)\,d\xi}{\int_B \rho(\xi)\cos\theta(\xi)\,d\xi} < \frac{\int_A \rho(\xi)\sin\theta(\xi)\,d\xi}{\int_A \rho(\xi)\cos\theta(\xi)\,d\xi}. \tag{4.27}$$

Since $\sin\theta(\xi) < \cos\theta(\xi)$ for all $\xi \in B$, and $\sin\theta(\xi) > \cos\theta(\xi)$ for all $\xi \in A$, the left term of (4.27) is strictly less than 1 and the right term of (4.27) is strictly greater than 1. This proves (4.27) and so provides the contradiction which establishes (4.24).

Claim 4.7. *For almost every $\xi \in (0,a)$, if $g_0(\xi) = c_a$ then $g_1(\xi) = c_a\sqrt{w^2(\xi) - 1}$ and, consequently, $(g_0(\xi), g_1(\xi))$ is the upper endpoint $(c_a w(\xi)\cos\psi(\xi), c_a w(\xi)\sin\psi(\xi))$ of V_ξ as defined in (4.18).*

Proof. This amounts to showing that the set

$$V := \{\xi \in (0,a)\colon\ g_0(\xi) = c_a,\ g_0^2(\xi) + g_1^2(\xi) < c_a^2 w^2(\xi)\}$$

has measure 0. If this is not true, then the function $u_1 := g_1\chi_{(0,1)\setminus V} + c_a\sqrt{w^2 - 1}\chi_V$ satisfies

$$\int_0^a u_1(\xi)\,d\xi > \int_0^a g_1(\xi)\,d\xi = 1. \tag{4.28}$$

Furthermore (in view of (4.4)) it is clear that $(g_0(\xi), u_1(\xi)) \in E_\xi$ for a.e. $\xi \in (0,a)$. Since $c_a > 1$ and $\int_0^a g_0(\xi)\,d\xi = a$ the set $V_1 = \{\xi \in (0,a)\colon\ g_0(\xi) < c_a\}$ must also have positive measure. Let V_* be some subset of V_1 which also has positive measure and define

$$\widetilde{g}_0 = g_0\chi_{(0,a)\setminus V_*} + c_a\chi_{V_*} \quad \text{and} \quad \widetilde{g}_1 = u_1\chi_{(0,a)\setminus V_*}.$$

Then $(\widetilde{g}_0(\xi), \widetilde{g}_1(\xi)) \in E_\xi$ for a.e. $\xi \in (0,a)$ and $\int_1^a \widetilde{g}_0(\xi)\,d\xi > \int_1^a g_0(\xi)\,d\xi = a$. If we choose the measure of V_* to be sufficiently small then we will also have, using (4.28), that $\int_1^a \widetilde{g}_1(\xi)\,d\xi > \int_1^a g_1(\xi)\,d\xi = 1$. Once again we can apply Claim 4.4 to obtain a contradiction. This proves that the set V has measure 0. □

Our next step is to show that

$$\text{the set } Q = \{\xi \in (0,a)\colon\ g_1(\xi) = 0,\ g_0(\xi) < c_a\} \text{ has measure } 0. \tag{4.29}$$

If this is false, then we consider the functions $\widetilde{g}_0 = \sqrt{\frac{1}{2}(g_0^2 + c_a^2)}\chi_Q + g_0\chi_{(0,a)\setminus Q}$ and $\widetilde{g}_1 = \min\{\sqrt{c_a^2 w^2 - \widetilde{g}_0^2}, c_a\}\chi_Q + g_1\chi_{(0,a)\setminus Q}$. It is clear that on the set Q we have $g_0 < \widetilde{g}_0 < c_a \leqslant c_a w$ and consequently also $\widetilde{g}_1 > 0 = g_1$. Consequently $\widetilde{g}_0$ and $\widetilde{g}_1$ satisfy (4.16). It is also clear that $(\widetilde{g}_0(\xi), \widetilde{g}_1(\xi)) \in E_\xi$ for a.e. $\xi \in (0,a)$. We can thus use Claim 4.4 to obtain a contradiction and complete the proof of (4.29).

Claim 4.8. *Suppose that, as in Lemma* 4.6*, the functions ρ and θ satisfy* (4.20)*. Then*

$$\rho(\xi) = c_a \min\left\{\frac{1}{\cos\theta(\xi)}, w(\xi)\right\} \quad \textit{for a.e. } \xi \in (0,a). \tag{4.30}$$

Proof. Let us use the notation $\widetilde{\rho}(\xi) = c_a \min\{\frac{1}{\cos\theta(\xi)}, w(\xi)\}$. In view of (4.25), it is clear that

$$(\widetilde{\rho}, \theta) \in \mathcal{P}_a \tag{4.31}$$

and that, furthermore, $\rho(\xi) \leqslant \widetilde{\rho}(\xi)$ for a.e. $\xi \in (0,a)$. Suppose, contrarily to what we claim, that the set $R = \{\xi \in (0,a)\colon\ \rho(\xi) < \widetilde{\rho}(\xi)\}$ has positive measure. Let us write $R = R_0 \cup R_1$ where $R_0 = R \cap \{\xi \in (0,a)\colon\ \theta(\xi) = 0\}$ and $R_1 = R \setminus R_0$. We observe that R_0 is exactly the set Q of (4.29) which has measure 0. Consequently R_1 has positive measure. This implies that the functions $\widetilde{g}_0 = \widetilde{\rho}\cos\theta$ and $\widetilde{g}_1 = \widetilde{g}_1 \sin\theta$ satisfy $\int_0^a \widetilde{g}_j(\xi)\,d\xi > \int_0^a g_j(\xi)\,d\xi$ for $j = 0, 1$. In view of (4.31) and Claim 4.4 this is impossible. □

We can now show that the functions ρ and θ which satisfy (4.20) also satisfy

$$\arccos\frac{1}{w(\xi)} \leqslant \theta(\xi) \leqslant \frac{\pi}{4} \quad \text{for a.e. } \xi \in (0,a). \tag{4.32}$$

In view of (4.25), we can do this by showing that the set

$$W = \left\{\xi \in (0,a)\colon\ \arccos\frac{1}{w(\xi)} > \theta(\xi)\right\}$$

has measure 0. Let us first observe that, by Claim 4.7, almost every $\xi \in (0,a)$ satisfying $g_0(\xi) = c_a$ also satisfies $\theta(\xi) = \psi(\xi) = \arccos\frac{1}{w(\xi)}$ and so is not in W. On the other hand, every $\xi \in W$ satisfies $\frac{1}{w(\xi)} < \cos\theta(\xi)$. Consequently, by (4.30), $\rho(\xi) = c_a/\cos w(\xi)$ or, equivalently, $g_0(\xi) = c_a$ for a.e. $\xi \in W$. So indeed W has measure 0 and we have proved (4.32).

Theorem 4.9. *Suppose that ρ and θ are the functions which satisfy* (4.20)*. Then $\theta(\xi)$ assumes a constant value a.e. on the set*

$$U = \left\{\xi \in (0,a)\colon\ \arccos\frac{1}{w(\xi)} < \theta(\xi)\right\}. \tag{4.33}$$

Proof. Suppose that the theorem is false. Then there exist two subsets A and B of U, each having positive measure, and numbers θ_0 and θ_1 such that $0 \leqslant \theta_0 < \theta_1 \leqslant \pi/4$ and

$$\theta(\xi) \leqslant \theta_0 \quad \text{for all } \xi \in A \quad \text{and} \quad \theta_1 \leqslant \theta(\xi) \quad \text{for all } \xi \in B.$$

We can assume further that each $\xi \in B$ also satisfies $\arccos\frac{1}{w(\xi)} < \theta(\xi) - \delta_0$ for some positive number δ_0, since, if not B can be replaced by some subset of positive measure which does have this property. Let $p = 1$ and let q be an arbitrary negative number. Then, if $\delta = \min\{\frac{\pi}{4} - \theta_0, \frac{\delta_0}{|q|}\}$, all the hypotheses of Lemma 4.6 are satisfied.

To complete the proof we will show that, for some choice of $q < 0$, both the inequalities

$$\int_A \rho(\xi)\sin\theta(\xi)\,d\xi + q\int_B \rho(\xi)\sin\theta(\xi)\,d\xi < 0 \tag{4.34}$$

and

$$\int_A \rho(\xi)\cos\theta(\xi)\,d\xi + q\int_B \rho(\xi)\cos\theta(\xi)\,d\xi > 0 \tag{4.35}$$

hold and thus we have a contradiction to the conclusion which would follow from Lemma 4.6.

We recall (cf. (4.19)) that $\rho(\xi) > 0$ for a.e. $\xi \in (0, a)$. So

$$\int_B \rho(\xi)\sin\theta(\xi)\,d\xi \geqslant \int_B \rho(\xi)\sin\theta_1\,d\xi = \sin\theta_1 \int_B \rho(\xi)\,d\xi > 0$$

and

$$\int_B \rho(\xi)\cos\theta(\xi)\,d\xi \geqslant \int_B \rho(\xi)\cos\frac{\pi}{4}\,d\xi = \frac{1}{\sqrt{2}}\int_B \rho(\xi)\,d\xi > 0.$$

Since $\tan\theta_0 < \tan\theta_1$ we have

$$\frac{\sin\theta_0}{\sin\theta_1} < \frac{\cos\theta_0}{\cos\theta_1}$$

and consequently the numbers

$$r_0 := \frac{\int_A \rho(\xi)\sin\theta(\xi)\,d\xi}{\int_B \rho(\xi)\sin\theta(\xi)\,d\xi} \quad \text{and} \quad r_1 := \frac{\int_A \rho(\xi)\cos\theta(\xi)\,d\xi}{\int_B \rho(\xi)\cos\theta(\xi)\,d\xi}$$

satisfy

$$r_0 \leqslant \frac{\int_A \rho(\xi)\sin\theta_0\,d\xi}{\int_B \rho(\xi)\sin\theta_1\,d\xi} < \frac{\int_A \rho(\xi)\cos\theta_0\,d\xi}{\int_B \rho(\xi)\cos\theta_1\,d\xi} \leqslant r_1.$$

Clearly every number q satisfying $r_0 < -q < r_1$ is negative and also satisfies (4.34) and (4.35). This completes the proof of the theorem. □

Let θ_a be the constant value assumed a.e. by $\theta(\xi)$ on the set U defined by (4.33). Then, perhaps after altering ρ and θ on sets of measure 0, we obtain that $U = \{\xi \in (0, a)\colon \arccos\frac{1}{w(\xi)} < \theta_a\}$. In view of (4.32), $\arccos\frac{1}{w(\xi)} = \theta(\xi)$ for a.e. $\xi \in (0, a)\backslash U$.

If $\theta_a = 0$, then U is empty and so $w(\xi)\cos\theta(\xi) = 1$ for a.e. $\xi \in (0, a)$. Consequently (cf. (4.30)) $\rho(\xi) = c_a/\cos\theta(\xi)$ for a.e. $\xi \in (0, a)$ and so

$$\int_0^a g_0(\xi)\,d\xi = \int_0^a \rho(\xi)\cos\theta(\xi)\,d\xi = c_a a.$$

But, since $c_a > 1$, this contradicts (4.8). We deduce that $\theta_a > 0$.

At the other extreme, if $\theta_a \geqslant \arccos\frac{1}{w(0)}$ then, since w is strictly decreasing on $[0, 1]$, we obtain that $U = (0, a)$ and it follows from (4.30) that $\rho(\xi) = c_a w(\xi)$ for a.e. $\xi \in (0, a)$. We also have

$$a = \frac{\int_0^a g_0(\xi)\,d\xi}{\int_0^a g_1(\xi)\,d\xi} = \frac{\int_0^a \rho(\xi)\cos\theta_a\,d\xi}{\int_0^a \rho(\xi)\sin\theta_a\,d\xi} = \tan\theta_a,$$

which implies that $\sin\theta_a = a/\sqrt{a^2+1}$. Consequently,

$$\int_0^a g_1(\xi)\,d\xi = \int_0^a \rho(\xi)\sin\theta_a\,d\xi = \frac{a}{\sqrt{a^2+1}}\int_0^a c_a w(\xi)\,d\xi. \tag{4.36}$$

In view of (4.2),

$$\int_0^a w(\xi)\,d\xi = -\int_0^a \frac{d}{d\xi}E(\xi, \alpha; \vec{X})\,d\xi = E(0, \alpha; \vec{X}) - E(a, \alpha; \vec{X}) = \sqrt{a^2+1}.$$

Combining this with (4.36) gives that $\int_0^a g_1(\xi)\,d\xi = ac_a$, which contradicts (4.8) and so establishes that $\theta_a < \arccos\frac{1}{w(0)}$.

From the past two paragraphs and the fact that w is strictly decreasing from $w(0)$ to 1 on $[0, 1]$ we deduce that there exists a unique number $\xi_a \in (0, 1)$ such that $\theta_a = \arccos\frac{1}{w(\xi_a)}$ and that $U = (\xi_a, a)$. This in turn implies that

$$\begin{aligned} a &= \int_0^a g_0(\xi)\,d\xi = \int_0^a \rho(\xi)\cos\theta(\xi)\,d\xi \\ &= \int_0^{\xi_a} \frac{c_a}{\cos\theta(\xi)}\cos\theta(\xi)\,d\xi + \int_{\xi_a}^a c_a w(\xi)\cos\theta_a\,d\xi \end{aligned}$$

$$\begin{aligned}
&= c_a\xi_a + c_a\cos\theta_a\big(E(\xi_a,\alpha;\vec{X}) - E(a,\alpha;\vec{X})\big)\\
&= c_a\xi_a + \frac{c_a}{w(\xi_a)}\sqrt{(a-\xi_a)^2+(1-\xi_a)^2}\\
&= c_a\xi_a + \frac{c_a}{a+1-2\xi_a}\big((a-\xi_a)^2+(1-\xi_a)^2\big)\\
&= \frac{c_a}{a+1-2\xi_a}\big(a\xi_a+\xi_a-2\xi_a^2+a^2-2a\xi_a+\xi_a^2+1-2\xi_a+\xi_a^2\big)\\
&= \frac{c_a}{a+1-2\xi_a}\big(a^2-a\xi_a+1-\xi_a\big).
\end{aligned}$$

So we have

$$c_a = \frac{a^2+a-2a\xi_a}{a^2-a\xi_a+1-\xi_a}. \tag{4.37}$$

We also have

$$\begin{aligned}
1 &= \int_0^a g_1(\xi)\,d\xi = \int_0^a \rho(\xi)\sin\theta(\xi)\,d\xi\\
&= \int_0^{\xi_a} \frac{c_a}{\cos\theta(\xi)}\sin\theta(\xi)\,d\xi + \int_{\xi_a}^a c_a w(\xi)\sin\theta_a\,d\xi\\
&= c_a\int_0^{\xi_a}\tan\theta(\xi)\,d\xi + c_a\sin\theta_a\big(E(\xi_a,\alpha;\vec{X}) - E(a,\alpha;\vec{X})\big)\\
&= c_a\int_0^{\xi_a}\sqrt{w^2(\xi)-1}\,d\xi + c_a\sqrt{1-\frac{1}{w^2(\xi_a)}}\sqrt{(a-\xi_a)^2+(1-\xi_a)^2}.
\end{aligned}$$

We have already calculated another expression for $w^2(\xi)-1$ in (4.14) and (4.15), so we can substitute it in both terms of the preceding line to get

$$\begin{aligned}
1 &= c_a\int_0^{\xi_a}\sqrt{\frac{2(a-\xi)(1-\xi)}{(a-\xi)^2+(1-\xi)^2}}\,d\xi + c_a\frac{1}{w(\xi_a)}\sqrt{\frac{2(a-\xi_a)(1-\xi_a)}{(a-\xi_a)^2+(1-\xi_a)^2}}\sqrt{(a-\xi_a)^2+(1-\xi_a)^2}\\
&= c_a\int_0^{\xi_a}\sqrt{\frac{2(a-\xi)(1-\xi)}{(a-\xi)^2+(1-\xi)^2}}\,d\xi + c_a\sqrt{\frac{2(a-\xi_a)(1-\xi_a)((a-\xi_a)^2+(1-\xi_a)^2)}{a+1-2\xi_a}}.
\end{aligned}$$

This latter formula can be rewritten as

$$\frac{1}{c_a} = \int_0^{\xi_a}\sqrt{\frac{2(a-\xi)(1-\xi)}{(a-\xi)^2+(1-\xi)^2}}\,d\xi + \sqrt{\frac{2(a-\xi_a)(1-\xi_a)}{a+1-2\xi_a}\cdot(a-\xi_a)^2+(1-\xi_a)^2}. \tag{4.38}$$

If we now substitute for c_a in this equation, using (4.37) we will obtain a rather complicated equation for ξ_a, which we will investigate further in the next subsection.

On a more simple level, we can use (4.37) to obtain estimates for c_a from above and below

$$\inf_{t\in(0,1)}\frac{a^2+a-2at}{a^2+1-(a+1)t} \leqslant c_a \leqslant \sup_{t\in(0,1)}\frac{a^2+a-2at}{a^2+1-(a+1)t}.$$

The function $t\mapsto\frac{a^2+a-2at}{a^2+1-(a+1)t}$ like any function of the form $A\frac{b-t}{c-t}$ where A, b and c are positive constants, is either an increasing or decreasing function on any interval which does not contain the point where its denominator vanishes. In this case, its minimum on $[0,1]$ equals 1 and is attained at $t=1$. Its maximum is $\frac{a^2+a}{a^2+1}$ and is attained at $t=0$. The maximum value of $\frac{a^2+a}{a^2+1}$ as a ranges over $[1,\infty)$ is attained at $a=1+\sqrt{2}$ and is thus equal to $\frac{4+3\sqrt{2}}{4+2\sqrt{2}}$ which is approximately equal to 1.2071.

4.4. Some numerical experiments

In this final subsection we present some numerical experiments, which lead us to a guess for the approximate value of the K-divisibility constant of $(\ell_2^2, \ell_2^\infty)$, namely $\gamma(\ell_2^2, \ell_2^\infty) \approx 1.0304$. Fix some value of a and try to find the corresponding value of $x = \xi_a$ by defining

$$f(x) = \int_0^x \sqrt{\frac{2(a-t)(1-t)}{(a-t)^2 + (1-t)^2}}\,dt + \sqrt{\frac{2(a-x)(1-x)((a-x)^2 + (1-x)^2)}{a+1-2x}} - \frac{a^2 - ax + 1 - x}{a^2 + a - 2ax}$$

and solving Eq. (4.38) which is simply $f(x) = 0$. We are using "Maple" via its interface with "Scientific Workplace". We will fix some values of a and then try to find $x \in (0,1)$ such that $f(x) = 0$. We are currently ignoring the question of whether such an x is unique. To find the corresponding value of c_a we compute $g(x) = \frac{a^2+a-2ax}{a^2-ax+1-x}$.

Here is a table which summarizes some of our numerical experiments, and which indicates that maybe the value of γ is approximately 1.0304:

a	x	c_a
1.2	.94667221295	1.0298
1.25	.94778089315	1.0304
1.275	.94811047015	1.0304
1.3	.94840470115	1.0304
1.5	.95139101435	1.0279
1.6	.95340037845	1.0259
1.8	.95781371025	1.0217
2	.96218058915	1.0179
2.2	.96618489325	1.0148
$1+\sqrt{2}$	.96997017725	1.0121
3	.977870722252	1.0073

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